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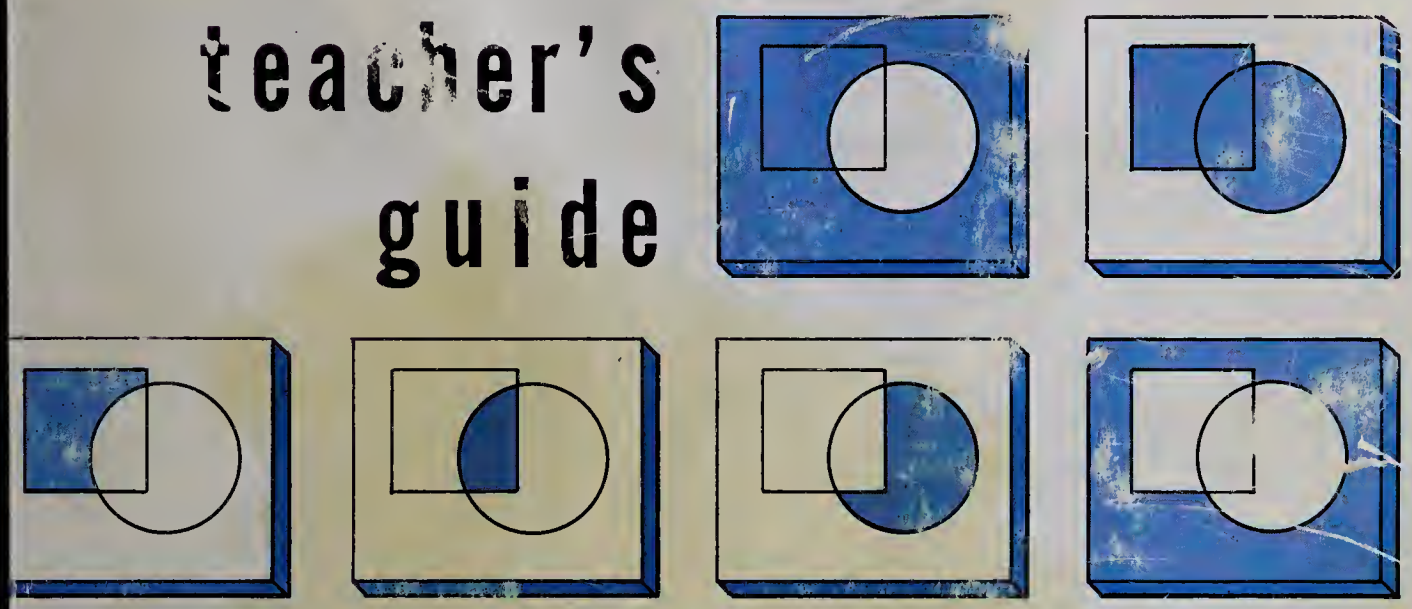
math workshop for children

teacher's guide



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ROBERT W. WIRTZ
MORTON BOTEL
W. W. SAWYER

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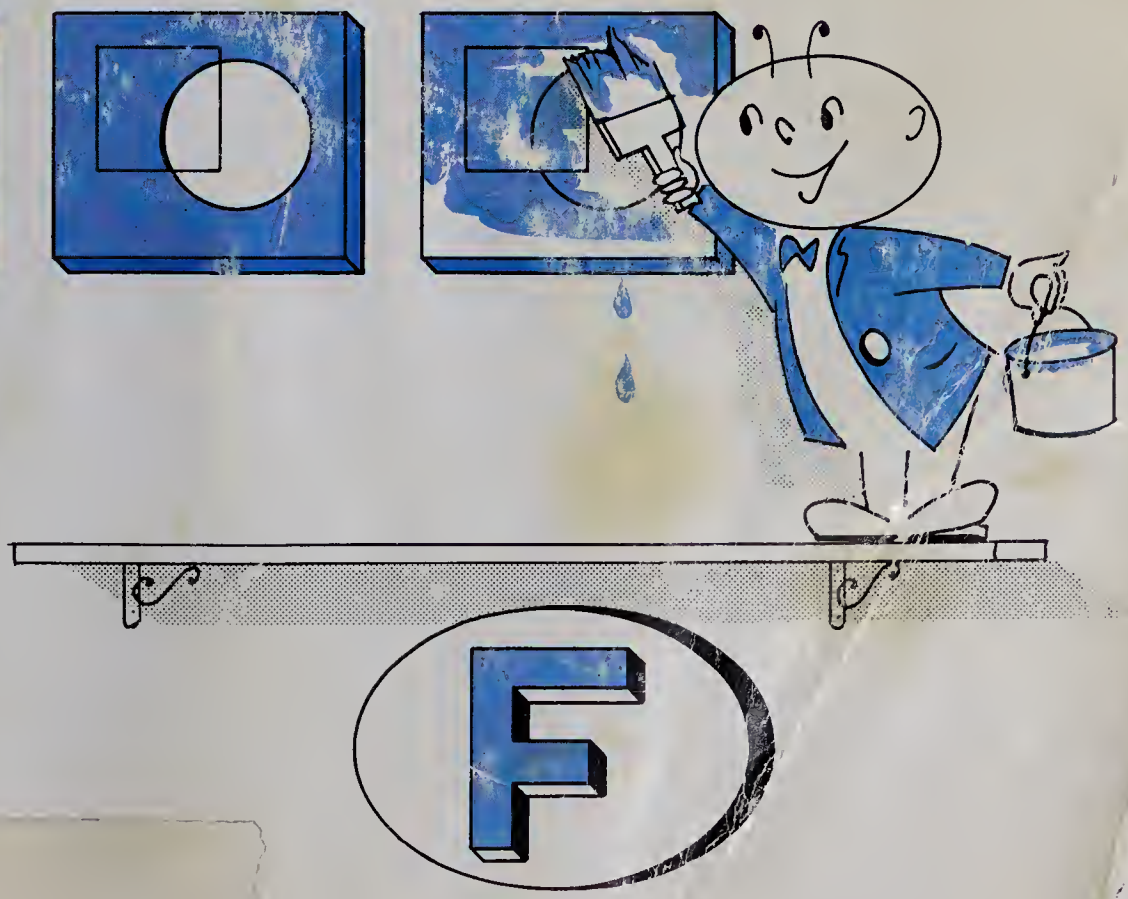

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LEVEL F
TEACHER'S GUIDE
MATH WORKSHOP FOR CHILDREN

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Preface

"Pattern Hunt" might well be taken as the theme for *Work Text Level F* of the *Math Workshop*. Some children and adults become overwhelmed by the details they encounter in mathematics. They reject mathematics as a collection of information so multitudinous that the memory is overwhelmed. They think of arithmetic as an area of endeavor in which the rule book is as full of entries as a dictionary, but without any equivalent of the dictionary's alphabetical organization.

The fact is, no one can memorize all there is to mathematics. The human mind is inadequate for the task and rejects such a waste of time. The only way to comprehend methematics is to consider it as a study of patterns—and it is the essence of the patterns that we remember, not their detail.

A four-year-old boy became fascinated with a clock. He learned the names of the symbols on the face—1, 2, 3, etc. He knew that 12 was at the top, 6 at the bottom, 3 on the right, 9 on the left; he knew that 7 and 8 were located between 6 and 9, etc.

One day, his father wrote the following for his son:

1 , 2 , 3 , 4 , 5 , 6 , 7 , 8 , 9 ,
10 , 11 , 12 , ___ , ___ , ___ , ___ , ___ , ___

"Thirteen comes next, but I don't know how to write it."

The boy looked at his father, and then back at the familiar labels on the clock. Then, all at once, he picked up his pencil and wrote "13" in the next blank. His eyes sparkled. His father smiled, and a big grin spread over his four-year-old's face.

With considerable speed now, the boy wrote 14 , 15 , 16 , 17 , 18 , 19 —and then hesitated. "How do you write 20?"

His father showed him, and added another line of blanks. The boy completed them without faltering, even when he came to 30. He continued writing all the way to 99. He had help with 100.

He knew the names assigned those symbols as far as 20. Now his interest turned to names for the other symbols he had written.

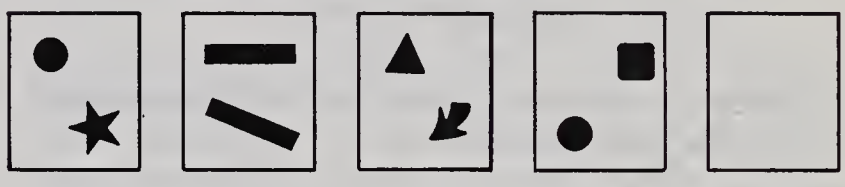
For days, he would tirelessly count from 1 to 100, smile at his accomplishment, and begin at 1 again. Eventually he wanted to know what came after 100.

As soon as he was told, he started counting 100, 101, 102, 103, 104, 105, 106—and then he stopped. He wasn't sure he wanted to count on beyond 106. So, he returned to his earlier love—counting from 1 to 100. Each time he would coast past the 100 to 106 or 107—and stop.

In a matter of a few minutes, this four-year-old child had grasped the pattern we use in writing the symbols from 1 to 100. After asking a few questions, he had fathomed the pattern we use in naming all those numbers.

Had he "memorized" the symbols and the names for 100 different numbers? Certainly not. He saw a pattern. That's all he needed. Could he describe the patterns? Certainly not, but he could demonstrate conclusively that they were now in his kit of tools.

A kindergarten teacher draws the following on the blackboard as the children watch:



She is about to fill in the last frame, but hesitates. She holds out the chalk. There are plenty of volunteers. So she draws several more empty frames until all have had an opportunity to get into the act. If someone fills in a frame that doesn't seem to fit, the group will be alive with helpers.

Very often, this kind of activity is carried on in utter silence. You can be sure, however, that each participant understands the idea. Each has seen a pattern he can extend. Later, children can be confronted with the following:

2	7	4	1		0	8		9	3
6		12	3	15			18		9

(Secret: Multiply the top row by 3 to get the bottom row) They will be eager to fill in the blanks. No explanation is necessary.

Then another variation:

6	16	10			36	24		49	46
2	4	5	3	7	9	6	5		
3	4	2	4	6			8	7	2

(Secret: Numbers in top row are products of factors written below them.)

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And another:

3	4	2	4	6			8	7	2
5	8	7			13	10		14	15
2	4	5	3	7	9	6	5		

(Secret: Numbers in center row are sums of addends written above and below.)

And a last one:

5	8	7				10		14	15
2	4	5	3	7		6	5		
3	4	2	4	6	9		8		
6	16	10			36			49	46

(Secret: Top row are sums of two addends immediately below each; bottom row are products of two factors written immediately above.)

"Secrets" are never revealed to children—and hints are rarely given.

The development outlined above was a pattern hunt. No words were needed. The patterns were simple, and once grasped, each child could quickly demonstrate that he had solved the riddle by filling in the blanks—thereby carefully completing what might have been intended as some much-needed drill.

From trivial beginnings, one hunt developed to include such questions as: "I'm thinking of two numbers; their sum is 15 and their product is 46. What are the numbers I thought of?" This is far from trivial.

Spiral curriculum

The concept of the spiral curriculum, as outlined by the *24th Yearbook of the National Council of Teachers of Mathematics*, has been used throughout all levels of the *Math Workshop*. At each level, from the first on, the main ideas of mathematics have been introduced. Each successive year's activities deal with these unifying ideas with increasing sophistication.

Less than mastery

However, children are not expected to "master" such elusive ideas as, say, multiplication of fractions or addition of positive and negative numbers—but they will meet all such ideas again and again, building always more layers of understanding.

The spiral curriculum rejects the "building-block" approach to arithmetic which has been used in the past to structure scope and sequence charts. Rather, the spiral curriculum calls for the early introduction of fundamental and unifying ideas, for repeated reconsideration at later times, continually nurturing the important concepts year after year as understanding develops.

The hope for "mastery" must be limited to that part of arithmetic that must be committed to memory—the vocabulary of arithmetic that must be instantly retrievable—the basic facts of arithmetic. Here, mastery can and must be achieved. But where big ideas are involved, mastery does not come in this way. It develops slowly and is based completely on understanding.

Approaches to individual activities

There are at least three generally successful ways in which individual pages or groups of pages can be handled in the classroom:

1) Use the *Work Text* to conclude a development;
2) use the page as the basis for discussion;
3) let children work independently.

1) In those areas in which the teacher has many ideas of his own, there is no substitute for direct teaching . . . with liberal use of manipulative material, the blackboard and class discussion. The *Work Text* is then used late in the development to tie down, or simply to record the progress that has been made to deeper understanding.

2) Other *Work Text* pages might, in contrast be gone through carefully, step-by-step, with the entire class—discussion encouraged.

3) A third approach might be called *Do-it-all-by-yourself*. Begin by saying, "Open your *Work Text* to (name the page). Look it over. Think about it. See if you can decide for yourself what is to be done. Wait a while before you ask for help. And let's make today a 'quiet day.'"

Now you feign reluctance to help but are, of course, watching carefully for those who haven't sufficient self-confidence to get started on their own. If several children are unable to get started, suggest that they work in pairs, helping each other quietly. Enlarge these helper combinations if necessary. Grouping like this will usually get the activity underway for everyone. Let time run out on the session if you wish, suggesting that those who haven't finished may want to take it home with them to finish.

Supplementary *Work Forms*

Many of the interesting and versatile page-forms developed in *Math Workshop* texts have been extracted and made available as Ditto and Multilith masters called *Work Forms*. In a few minutes, a teacher can convert one of these forms with pen or pencil into a rich supplementary activity for the whole class or for a group within the class. Providing, as they do, significant, independent learning activities for creative seatwork and meaningful homework, and taking into consideration, as they do, the wide range of individual abilities within any class, the *Work Forms* should prove very useful to busy teachers.

The 59 masters include:

10 Number Line Forms, numbered N1 to N10
14 Cross Number Forms, numbered C1 to C14
35 General Forms, numbered G1 to G35

The *Work Text* itself suggests many ways in which these forms can be used to help develop important ideas (and, at the same time, provide essential drill in the fundamental facts). Teachers can create similar activities by adding whatever numbers they want in a matter of minutes.

Countless other uses for the *Work Forms* can be found in the children's work texts and the teacher's guides to the other levels of the *Math Workshop*. The ideas in the particular adaptations found there are easily adaptable to any level of development.

Mastery Tests

The set of 59 *Work Forms* contains a battery of six tests, each of which presents all the basic facts of addition, subtraction, multiplication and division. An answer sheet is provided so that children can check their own or their neighbor's work. In addition, an individual record sheet enables each child to chart his own progress in terms of accuracy and time required to recall all the basic facts.

The *Work Forms*, then, provide the teacher with a flexibility and versatility never before available in an organized and integrated program. While they are not essential to the use of the *Math Workshop* texts, they are an invaluable supplement, an effective means of meeting individual needs, and a help to every teacher in expressing his own individual approach to helping children understand arithmetic.

Problem solving

Many people use the expressions "problem solving," "word problems," and "story problems" interchangeably in referring to such written questions as "Johnny lost 7 of his 11 pennies. How many has he left?"

Another example: "Farmer Brown had 8 posts. Each weighed 10 pounds. How much did the posts weigh altogether?" The answer, of course, is 80 pounds because $8 \times 10 = 80$ . . . or is it $10 \times 8 = 80$?

A slight change in the statement transforms it into a bit of a problem—into more than an example. "Farmer Brown had 8 posts. He set them out in a straight line, and 10 feet apart. How far is it between the end posts?"

To answer the question, you had better not look for "cue" words. Think for a moment; visualize or make a sketch of the situation. Once you understand the essence of the problem, then only is it reduced to an example from life of the pattern that "$7 \times 10 = 70$."

Now, both "examples" and "problems" have a vital role in helping children understand ways in which arithmetic applies in the physical world, but let us explore the use of "examples" in the teaching of elementary school arithmetic.

Professor W. W. Sawyer tells about a class of able fourth grade students that was asked: "Where would you ever use the fact that $24 \div 4 = 6$?" After considerable thought, a boy replied: "I might use it to answer some questions at school." "Yes," Professor Sawyer admitted, "but where else?" "Well," another student said after deep thought, "My mother might ask me to go to the store and get her $24 \div 4$ eggs."

"My mother never did that to me," was Professor Sawyer's reply.

Such episodes suggest a school program in which "examples" are not used as effective tools to help develop better understanding of arithmetic.

"Examples," when properly used, can be, so to say, the first baseman who clinches a very effective triple-play: from manipulative materials-to blackboard-to examples . . . examples from life.

The play begins in the world of objects children can feel and see and manipulate. A pattern is noted in otherwise unrelated events. Sketches and diagrams on the blackboard can help focus attention on the aspect of the experience that has been singled out for consideration. We need to be able to refer to a pattern we have noticed, so we attach a symbol or group of symbols as a name for the pat-

tern and thereby increase our command of the new language of arithmetic. This is an act of abstraction. We have given a name to a pattern.

Can each child use this new bit of language? Can he tell us about examples in which he recognizes this same pattern?

"Tell us a story. . . ." If the child can use his new language in this way, we've accomplished our immediate goal. If he can't, we must go back to the world of objects at hand and try again.

The child's response at the vital story-telling stage is the most reliable evidence a teacher can have that a satisfactory level of understanding has been reached. Without this immediately available evidence, the teacher could unwittingly leave some children lost in the strange world of abstract symbols—and that, for a child, is a chilling experience.

Children are brought to school by proud parents who tell us their Mary can count all the way to 25. But when she counts the fingers on her two hands, she ends up with the surprising conclusion that she has 13 fingers!

Other children are described as already knowing the addition facts. See for yourself: "Billie, how much is 4 plus 3?" He performs with an appropriate "7." But later we talk to Billie about a party where four children were sitting at a table, and then three more children arrived. "How many were there then?" He doesn't know.

So every good teacher knows he must find out what the language of arithmetic a child uses actually means to the child.

Most children, by this time, have learned that $4 \times 6 = 24$. They know that in 4 rows of checkers with 6 in a row, there are 24 checkers. If necessary, you can so arrange checkers or blocks or tongue depressors and carry out the counting process. And children would be sure that the same number of checkers could be arranged in 6 rows with 4 in each row. Further, they would realize that "4 columns with 6 in each" and "6 columns with 4 in each" are also descriptions of the arrangement. They would understand that "4 rows and 6 columns" and "6 rows and 4 columns" also refer to the same kinds of arrangements of 24 items of any kind.

This is the first part of the triple play . . . a recognition of a particular pattern in the world of things. Next, this pattern needs a name. A sensible name would be:

$$4 \times 6 = 24$$

This "naming" is the second part of the triple play. It has taken us out of the world of things and we are in the world of abstraction. We have written a series of symbols and have agreed that these symbols point clearly to the pattern we noted in the physical world.

But, let's hurry back to the more familiar world of things. "Think of that name for a pattern as a *headline.* Where might you encounter an example of that pattern?" Or, "Make up a story that fits that headline." Whether the stories are from the world of real things or the child's own world of imagination makes little difference—as long as those worlds operate in such a way that $4 \times 6 = 24$ in the same sense that an array of 4 rows and 6 columns of checkers contains 24 checkers.

"It's 6 blocks to school and I make two round-trips a day, so I walk 24 blocks." This story clearly fits the headline, even though 4 comes in only indirectly—because the child assumes that we understand that four one-way trips and two round-trips are the same distance.

"Twenty-four men went to the moon in four spaceships. Each ship has two pilots and a crew of four. There were six in each spaceship." The last sentence was added, we suppose, for those who want to know where 6 comes into the situation.

"There are 6 gallons of milk in 24 quarts." Does that story fit the headline? Does it point to a pattern that has something in common with that aspect of the array of checkers we named "$4 \times 6 = 24$?" Certainly. Imagine six gallon cartons of milk in a row. Now each carton is replaced by a column of four quart cartons. We now have 4 rows of cartons with 6 in a row—4 rows and 6 columns.

This is the third part of the triple play—finding a few of the unlimited number of "examples" from actual or imagined experiences which display the pattern we are discussing.

Thus, the search for examples is an essential part of the developmental work in the classroom. The children are the searchers. They are deepening their understanding of the relationships that exist between the abstract world of mathematics and the world we know through our senses, the physical world. This is a creative experience.

In such activities, "examples" play a fundamental role in the child's growth in comprehension of the meaning of mathematics.

In contrast, contrived "story problems" in text-

books, often used as tests, lead to a distortion of the relationship between mathematics and the real world—it leads children to say, "My mother might send me to the store for 24 ÷ 4 eggs."

Math Workshop work texts contain very few typical "word problems." However, in the teacher's guides, ample suggestions are given to help teachers stimulate all children in fruitful searches for examples of the patterns of arithmetic the children are considering.

Headlines and Fact Teams

We can think of no useful end served by arguing that one member of a fact team is a more appropriate "headline" for an example than are the other members of the team. Consider this team:

$4 \times 6 = 24$ $6 \times 4 = 24$ $24 \div 6 = 4$ $24 \div 4 = 6$

Our main emphasis here is that all these members are appropriate names for a pattern we see repeated over and over again in the real world. The focus is on the pattern, not on the particular expression that points to the pattern.

Arrange 24 checkers in 6 rows and 4 columns. That arrangement of things has something about it worth noticing. Any member of the fact team points to the pattern. No team member has any claim to being *most* appropriate.

The headlines, $24 \div 4 = 6$ and $4 \times 6 = 24$, are simply two different ways of pointing to a particular pattern. Some would argue that, "Six children confronted with the task of dividing 24¢ equally" ought to be analyzed in such fashion that "$24 \div \_\_\_\_ = 6$" is a proper statement of the problem, and "4¢" the proper solution. This centers the attention on discussion about "What is happening? What are the groups? Can you divide cents by people?"

The "understanding" in this approach is directed toward "understanding rules" to justify the argument that $24 \div 4 = 6$ is correct and that $24 \div 6 = 4$ in this instance is wrong. This must be terribly confusing to children and teachers—and parents.

Shift the concern to the pattern involved. What pattern must you recognize to help divide 24¢ equally among 6 children? The pattern we need to see is the same pattern we saw in the checkers or milk cartons. We named it with a fact team and each member of the team simply calls to mind the pattern!

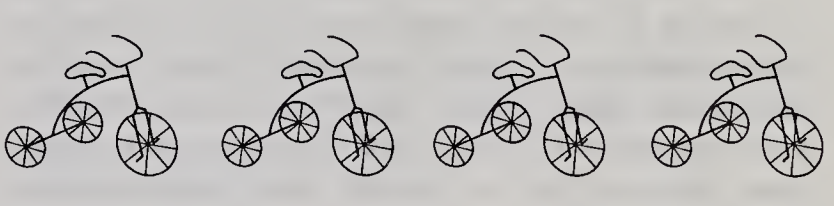
Suppose a child argues: "Half the children will get half the money—or 12¢ for 3 children. So, each child will get a third of 12¢—or 4¢ for each." So he suggests the headline "$(24 \div 2) \times 1/3 = 4$" as appropriate. Is he wrong? We hope not! . . . We hope that $(24 \div 2) \times 1/3 = 24 \div 6$.

Our concern is that children understand patterns and the relationships between the patterns of mathematics and the patterns of the world of experience. We hope they do not need rules to help them conclude that "each child gets 4¢."

Put the following incomplete headlines on the board. Ask the children to complete each so that it is a true statement and then make up a story that "fits the headline" in each case.

A.	$4 - 2 = \_\_\_\_$	B.	$3 \times 4 = \_\_\_\_$
C.	$\_\_\_\_ = 2 + 3$	D.	$12 = 4 + 4 + 4$
E.	$4 \times \_\_\_\_ = 12$	F.	$15 - 3 = \_\_\_\_$
G.	$\_\_\_\_ + 3 = 9$	H.	$\_\_\_\_ \div 4 = 3$
I.	$12 \div \_\_\_\_ = 4$	J.	$8 + 4 = \_\_\_\_$

Several of the headlines seem to point to the same pattern. Which ones? (Candidates will be B, D, E, H, I; and possibly an argument will be advanced for F.) How many fit the following sketch if we direct our attention to "wheels?"



"Finish all the proposed headlines."

$3 \times 4 = 12$ $4 \times 3 = 12$ $12 \quad 3 = 4$

$12 = 4 + 4 + 4$ $12 \quad 4 = 3$ $8 + 4 = 12$

Shall we accept all these headlines as appropriate? Does each point to a pattern that fits the real-life situation we are confronting? The easy way out would be to accept them all as different ways of saying something about the "number of wheels" in this case. That's the easy way . . . and why not take it?

Especially since the alternative has a gloomy outlook to it. We might soon be involved in a philosophical argument that would find us arguing that 3×4 does not equal 4×3. At least, we would have to spin a whole set of confusing rules so that, in light of those rules, we could say that one candidate's headline complies more fully than another. The result of such a contrived obstacle is to make "problem solving" a distasteful, complicated, lifeless effort to remember rules and recognize cue words that will help us recall from memory the right rules so we can come to a correct answer.

Any of the headlines we are considering is but another way of pointing to the pattern of the "number of wheels." As soon as a child has completed any one of the headlines that he thinks appropriate, he is in command of a fact of arithmetic that applies. How does he apply it? . . . if by "rules," he is apt to make a mistake; if by thinking about the situation, he will make a sensible reply.

But this part of the activity—recognizing that there are 12 wheels on the 4 tricycles—is only the initiating and less important part of a developmental situation.

"Now," we say, "think up other examples or stories that fit all those headlines." This leads us toward a creative activity in which children explore the relationship between arithmetic and the real world.

"Three chairs have 12 legs."

"There are 12 legs on 4 three-legged stools."

"In 3 rows of trees with 4 trees in a row, there are 12 trees."

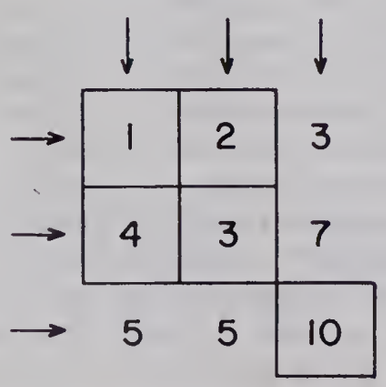
"On 3 wagons, there are 12 wheels."

"Two round-trips to Grandmother's is 12 miles, so she must live 3 miles away."

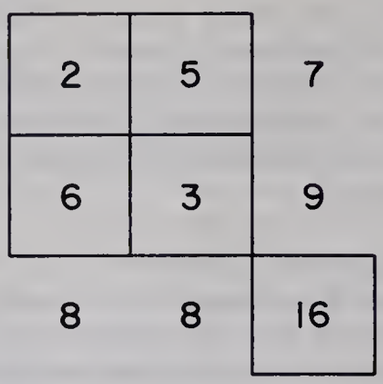
There is a common aspect about all these more or less seemingly diverse situations drawn from the real world. Any of the headlines above point to that common aspect. Any of the completed headlines contains the arithmetic each statement or example relies on.

We ought always to simplify the relationship between arithmetic and the physical world around us. We ought to worry when anyone tries to complicate and propose endless, useless, dangerous rules.

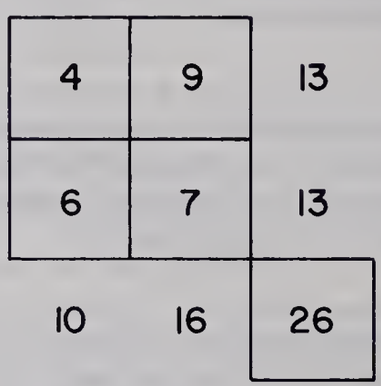
Below is a Cross Number Puzzle form. The problem is to select a number for each of the 4 cells so that if we add across each of the 2 rows and then add their sums, the grand total can serve equally well as the grand sum if we go on to add each of the two columns and total our results. Here is one group of 4 numbers that meets the test:



Perhaps you notice that the sum of each column is 5. Try another group of 4 numbers with the same pattern in the two columns—the sum is 8 in each.



Try an example in which the columns do not follow the pattern we notice:

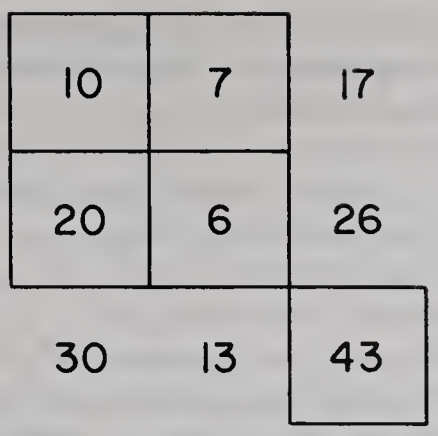


Now the rows add up to the same sum. There's a good bit of symmetry evident and some feel quite confident that this is a scheme that will keep on producing solutions to this puzzle. And it will. But the whole truth is much more startling.

We have explored this idea in the abstract manipulating numbers, inspecting sums, and sums of sums. But, bring it down to the world of things—cows, this time?

Farmer Brown put four cows in one field, nine in another, six in another and seven in the last field. In the two north fields, there are 13 cows and in the two south fields, there are 13 cows—or 26 in all.

Wouldn't it be surprising if there hadn't been the same number of cows regardless of the route over which he drove them? If we came out with any other result, we would think something unnatural had happened along the way. So, we are led to suspect that our puzzle will "work" regardless of what number of cows Mr. Brown puts in those 4 fields.



How clear the whole matter becomes when we take it out of the abstract and find a suitably similar example in the world of objects.

Incidentally, do you notice how quietly an example of "regrouping" and "carrying" was insinuated? If we want to add 17 cows and 26 cows, separate them into fields so that in one column of fields the numbers of cows are multiples of 10.

Headlines and "What's My Rule?"

Now, let's play "What's My Rule?" At first, teacher is "it." She asks the child to call out a number, such as 3. She then "applies her rule" and answers with "6." This example is recorded:

$3 \rightarrow 6$

As children call out numbers, and the teacher applies her rule, results accumulate:

$3 \rightarrow 6$
$5 \rightarrow 10$
$4 \rightarrow$

"I know what you are going to answer," Mary blurts out. "You are going to write 8."

Mary takes over from the teacher. The children keep calling out numbers, and Mary "applies the rule" in responding.

$4 \rightarrow 8$
$1 \rightarrow 2$
$2 \rightarrow$

"You're going to answer by saying 4!" By this time, many of the children think they know the rule.

"Now, let's use these results as a headline. What experience could we be talking about? We could be telling you how far we live from school, and then you tell us how many blocks in a round trip. If I say 2 because I live two blocks from school, you'll answer 4 because that's how far I would walk to school and back. Or maybe Mary and Dick put their books in two piles with the same number in each. The first number tells how big the piles were, and then you tell how many books they started with."

Or, "My father told me he would give me as much as I save from my allowance each week. If I save 4¢, he gives me 4¢ extra and I have 8¢ to put in the bank." And so forth, until the teacher tells the class she has another rule. The list of results accumulates like this:

$1 \rightarrow 8$
$5 \rightarrow 10$
$3 \rightarrow 8$
$0 \rightarrow$

"Five," Bill says firmly.

"If I say 2, you're going to say 7," someone else ventures.

When most children have indicated their grasp of the rule, the results become a headline. What story could fit that headline?

"A boy found a nickel. The first number tells how much he had to start with and the other number tells how much he had after he found the nickel."

"After we drove a ways, we had five miles left to go. If we had gone one mile, then we have five more to go and the whole trip is six miles."

In another "What's My Rule?" game, the teacher asks for a pair of numbers. She rejects some pairs because they don't fit her rule. The following headline is the result:

$3,1 \rightarrow 2$
$5,7 \rightarrow 6$
$6,2 \rightarrow 4$
$0,6 \rightarrow 3$
$4,2 \rightarrow$

"You're going to write 3 . . . and I know a story. "Bill had six books and his sister had two. They put them together and then divided them evenly—each got four."

"In my closet there were three shoes on the shelf and one on the floor. You know now that I have two pairs of shoes."

Another game produces this headline:

$5,7 \rightarrow 5$
$10,9 \rightarrow 9$
$8,5 \rightarrow 5$
$8,6 \rightarrow$

"You are going to answer '6' . . . and your rule is like this. Popsicles are 8¢ at one store and 6¢ at another. You can be sure I only spent 6¢."

So the game moves constantly back and forth

between abstractions of arithmetic and similar patterns in real-life situations. Before we leave each example, every child in the room (almost without exception) discovers the secret rule for himself, if not in the list of abstract numbers, then in the examples from life.

These simple games that involve all children included the idea of multiplication, addition, division, average and inequalities. At no time is any child asked to *verbalize* the "rule." If he could anticipate the result while the headline was being constructed, he was operating on a higher level of discovery than those who saw what was going on only as the examples from experience developed. In either case, the children were moving back and forth between the numbers of arithmetic and their use in the real world.

It is obvious that this game can deal with an unlimited variety of relationships on any level of maturity. Each child is challenged to participate on the highest level for which he is equipped by his own talent and experience.

The headline writer sometimes invents shorthand that seems strange, at first. The story writer needs some clues. Here is such a headline, and a group of hints:

$$\begin{array}{r} 1,2 \\ +0,3 \\ \hline , \end{array} \qquad \text{hints:} \qquad \begin{array}{r} 0,2 \\ +0,1 \\ \hline 0,3 \end{array} \quad \begin{array}{r} 0,3 \\ +0,2 \\ \hline 1,0 \end{array} \quad \begin{array}{r} 1,3 \\ +0,1 \\ \hline 1,4 \end{array} \quad \begin{array}{r} 0,4 \\ +1,1 \\ \hline 2,0 \end{array} \quad \begin{array}{r} 0,3 \\ 0,3 \\ \hline , \end{array}$$

Are there enough hints? Suppose there aren't, and the teacher must tell about the boy who first held up all the fingers on one hand and two on the other. Then he raised three more fingers. Now he held up all the fingers on two hands. So we complete the headline this way:

$$\begin{array}{r} 1,2 \\ +0,3 \\ \hline 2,0 \end{array}$$

Can you tell a version of the story that fits this headline? Probably, but again suppose we haven't much help from the class. So we say, "A girl had one nickel and two pennies. She earned three more pennies and wanted to change them so she had the fewest coins. She had one nickel and five pennies. She changed the five pennies into a nickel and then had only two coins—both nickels." The light begins to dawn. Let's try one of the hints.

$$\begin{array}{r} 0,3 \\ +0,2 \\ \hline 1,0 \end{array}$$

"Bill had three pennies in one pocket and two in another. At the bank he changed them for a nickel and had no pennies left."

"I had three fingers up and then put up two more. Then I had one handful of fingers up." But, someone argues, maybe he had three on one hand and two on the other. To fit the headline, we need a rule of some kind—like the "fewest coins" rule—perhaps we require that you must always fill out a handful of fingers if you can. Then the incomplete hint:

$$\begin{array}{r} 0,3 \\ +0,3 \\ \hline , \end{array}$$

Complete this headline and give an example that fits it.

"Three pennies and three pennies equals six pennies. Five of them can be changed to one nickel and there is one penny left or one nickel and one penny in all," (The headline 0,3 + 0,3 = 1, 1 can be read "zero comma three plus zero comma three equals one comma one.")

"We have three school days this week and three school days next week. That's the same as one week of school days and one more day."

$$\begin{array}{r} 1,0 \\ -0,3 \\ \hline , \end{array}$$

"I had a nickel and spent 3¢. That left me with no nickel and two pennies."

"I put down three out of a handful of fingers. There were only two left up."

$$\begin{array}{r} 2,2 \\ -\ 1,4 \\ \hline 0,3 \end{array}$$

A completed headline often stimulates a greater variety of responses—if such are encouraged.

"I have one nickel and four pennies left after spending 3¢. I started out with two nickels and two pennies."

"I can't put up three more fingers if I already have one handful up and four up on the other hand. I would be two fingers short."

$$\begin{array}{r} 1,2 \\ \times\ \ 3 \\ \hline , \end{array}$$

"Each of three girls had one nickel and two pennies. They put their money together and had three nickels and six pennies. They changed five of the pennies for a nickel and then had four nickels and one penny." And, eventually:

$$3\overline{)4{,}1}\quad 1{,}2$$

"Bill had three nickels and a penny. He spent a third of it. That is, one nickel and two pennies. Four nickels and one penny is the same as three nickels and six pennies. One third of each is one nickel and two pennies."

When the children have the idea worked out, a new headline is considered and obviously requires some new hints.

0,8		0,5	0,10	1,5	1,6
+ 0,4	hints:	+ 0,6	+ 0, 4	+ 0,5	+ 1,6
1,0		0,11	1, 2	1,10	

"There were eight eggs in one carton and four eggs in another. All could be put in one carton of one dozen—and none left over."

"The little hand of the clock has gone around to 8. In four more hours it will have made exactly one trip around the clock."

"A board 8 inches long and one 4 inches long can be cut out of a 1-foot board."

"My last birthday was eight months ago. My next birthday will be in four more months because eight months and four months are one year and no months."

Then there is the incomplete hint above—

$$\begin{array}{r} 1{,}6 \\ +\ 1{,}6 \\ \hline \end{array}$$

"A dozen and a half plus a dozen and a half is three dozen."

"Out of 3 feet of ribbon, I can cut two pieces, each 1 foot and 6 inches long."

All these headlines have a bit of a puzzle to them. The notation is strange, but it helps express clearly many events in the real world. Part of the purpose in providing children with such activities is exercise in using what some people call *denominate numbers*—adding 8 inches to 1 foot and 6 inches, and then changing 1 foot and 14 inches into 2 feet and 2 inches. But there is more.

Consider these headlines.

1,7		0,3	0,5	3,6
+ 0,6	hints:	+ 0,7	+ 1,6	+ 1,6
		1,0	2,1	5,2

"Jim had one dime and seven pennies. He earned six more pennies. He changed his dime and thirteen pennies into two dimes and three pennies."

$$\begin{array}{r} 3{,}2 \\ -\ 1{,}5 \\ \hline \end{array}$$

"Mary spent a dime for a candy bar and 5¢ for gum. She started out with three dimes and two pennies. She had one dime and seven pennies left."

$$\begin{array}{r} 1{,}4 \\ \times\ \ 3 \\ \hline \end{array}$$

"If each of three boys took a dime and four pennies to the store to make a purchase, they had three dimes and twelve pennies. First they changed their money into four dimes and two pennies."

One of the important dimensions to this activity is now apparent. In this final series of examples, we find that we no longer need the commas in the notation.

$$\begin{array}{r} 23 \\ -\ \ 8 \\ \hline \end{array}$$

"I had 23¢ and spent 8¢. I had one dime and five cents left, or 15¢."

On our somewhat circuitous route through hands and dozens and dimes, we have been exploring the idea of place value; and, we have been carrying and borrowing without staying long at any time in the abstract world of numbers. Every step we took was based solidly on the physical world of things.

This *headline* technique can be used to help develop every idea of arithmetic on every level of development. It's an old idea that is familiar to all teachers. All teachers have, from time to time, asked children to provide examples in which facts of arithmetic are reflected in life. But, this activity ought to become a theme permeating daily activity.

Examples called "story problems" have usually been reserved for use as tests of what children remember about trips through the abstract reaches of number manipulation. "Now that the class has learned a new group of division facts," we say, "let's see if they can use this new knowledge."

Children can be creators of their own examples—answers and all—and some of theirs will be much more difficult than any that could be included in a textbook. Since most of the creative activity depends on the spoken language, there is less of a reading problem to slow down the arithmetic. Reliance is on the speaking vocabulary which is usually several years ahead of the reading vocabulary.

Children are also given written examples. They must work out the appropriate headline, but that's only the beginning. As soon as they have found and correctly interpreted a headline, they are sent scurrying about for many other examples that would fit the same headline. In this way, usually trivial story problems launch a spirited search for suitable and more personally satisfying examples. Many children demonstrate remarkable ingenuity in creating a fascinating example to fit a headline that started from the question, "How many children are there in two sets of triplets?" You're apt to be out in space in no time at all, or deeply involved in a baseball game, with "two sets of outs" each "inning."

This activity obviously provides adequately for individual differences. There is opportunity for each child to participate on his own level—and to enjoy success.

Each pupil is urged to stretch his efforts in the direction of increased creativity.

Because the acitivities outlined here never lose intimate touch with the physical world, children can move on at an unexpected speed—and with complete understanding. Teachers become the thought-starters, the probers, the leaders who open to children the exciting world of mathematics. Once the interest in pattern and the search for its extension has begun, children are anxious to move on at their own speed. A breath of freedom and sense of adventure enters arithmetic.

If the method proposed here as a main classroom technique has been made to stand on its own feet, then the time has come to give away a secret. The secret was saved for the end—after all the arguments were in—so that skeptics would keep on reading.

The secret is a simple one and one easily demonstrated in any classroom. It is that children actually enjoy looking for their own examples in their own world of fact and fancy. Where this hunt for suitable illustrations has been used consistently in classrooms, it has never failed to excite keen interest in arithmetic. Children have good, old-fashioned fun as they deepen their mathematical understanding.

Other ideas from other levels

Many activities suggested for other levels of the *Math Workshop* are easily adaptable to many levels of maturity. Some are reviewed here as aids to creative classroom work that are rich in mathematical content, stimulating for all, and open-ended to provide for individual differences.

1. The language of addition and multiplication

In arithmetic, there are two processes that make it possible to *operate on* or think about a pair of numbers in order to suggest a particular third number. These basic processes are called addition and multiplication.

Addition or multiplication, and their inverses, subtraction or division, can be discussed in a few meaningful words. The key words used in addition and subtraction are: *two addends and a sum.* Addition and subtraction boil down to a search for a missing term: an addend or a sum, when two of the three terms are given.

	MISSING	GIVEN		
		Addend	Addend	Sum
2+3 = ___	sum	✓	✓	
1+___=8	addend	✓		✓
___+7 = 10	addend		✓	✓
7−3 = ___	addend		✓	✓
7−___ = 2	addend	✓		✓
___−10 = 15	sum	✓	✓	

The key words used in multiplication and division are: *two factors and a product.* Multiplication and division boil down to a search for a missing term: a factor or a product, when two of the three terms are given.

	MISSING	GIVEN		
		Factor	Factor	Product
3×4 = ___	product	✓	✓	
___×7 = 21	factor		✓	✓
5×___ = 30	factor	✓		✓
24÷3 = ___	factor		✓	✓
18÷___ = 9	factor		✓	✓
___÷6 = 7	product		✓	

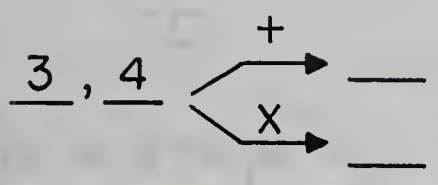
Main credit for this greatly simplified language belongs to the School Mathematics Study Group (SMSG). Instead of preserving the confusing vocabu-

lary that has been traditionally repeated in elementary arithmetic textbooks, SMSG has chosen to reflect the way mathematicians actually think and work. Looking at the operations of arithmetic in this way makes it possible for a pupil to have confidence enough to say, *If I can add, of course I can subtract, and if I can multiply, of course I can divide.*

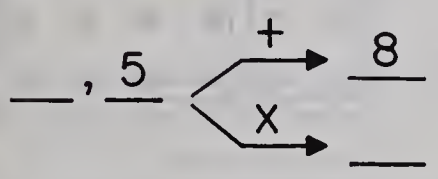
2. Basic fact practice games

When a pair of numbers is considered as addends and as factors that point to a sum and a product, four numbers are involved—the pair, a sum, and a product. *If any two are given, can you find the other two?* (Yes, unless 0 is given as one of the pair and as the product.)

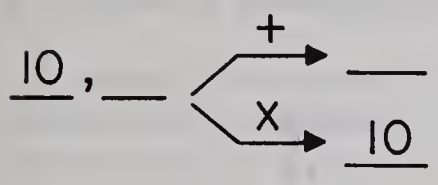
There are many ways to ask these questions. Three suggestions are given below:



(Answer: Sum must be 7 and product must be 12)



(Answer: Missing member of pair is 3 and missing product must be 15)



(Answer: Missing member of pair must be 1 and missing sum must be 11)

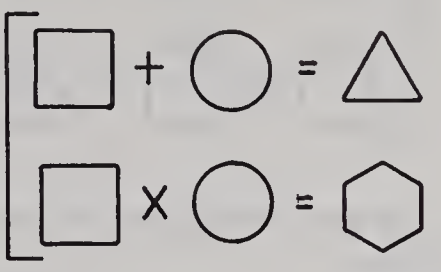
In another form, the questions may be asked this way:

	Product	Pair	Sum
A		3,2	
B	16		10
C	13	1,	
D		3,	6
E	7		

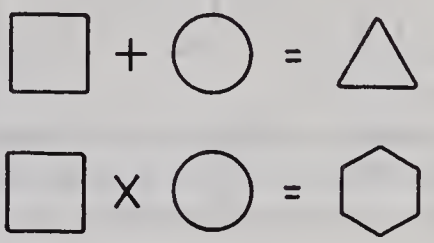
(Missing product is 6 and missing sum is 5)
(Missing pair must be 2,8 or 8,2)
(Missing member of pair must be 13 and sum must be 14)
(Missing member of pair must be 3 and product must be 9)
(Missing pair must be 1,7 or 7,1 and sum must be 8)

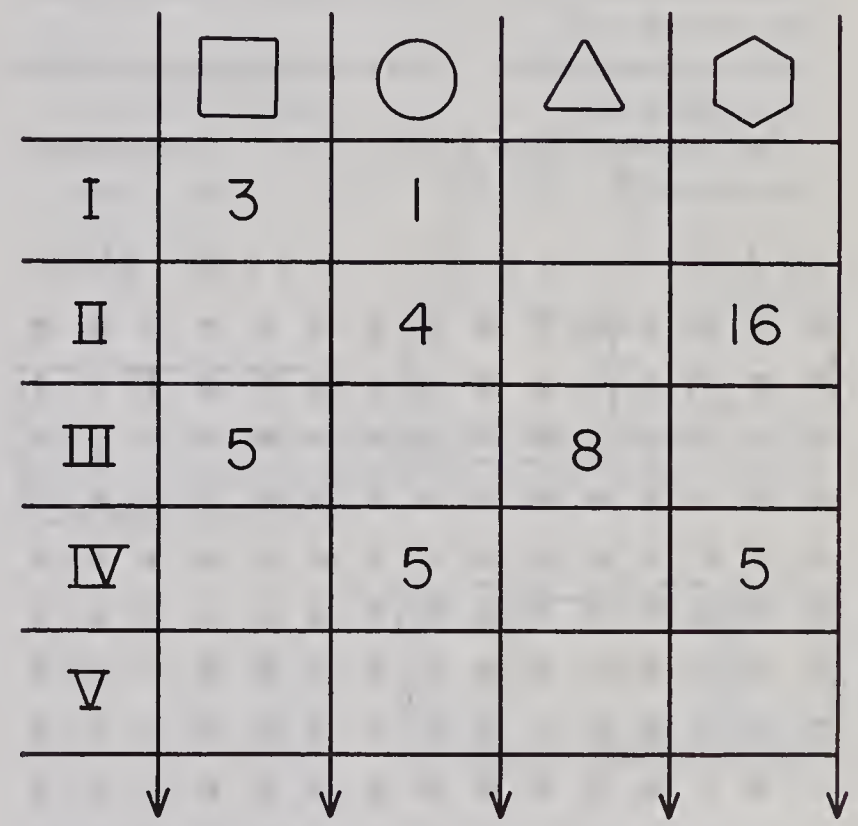
(When the product is a prime number, and it is given, then the pair must be composed of that number and one—the only factors of a prime number.)

In yet another form, the questions may look like this:



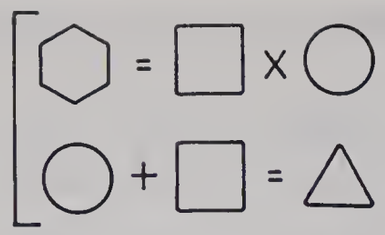
Now, an agreement is in order. In each example, any number that is indicated for a shape must occur in all similar shape's. The rules do not prevent the same number occurring in different shapes.



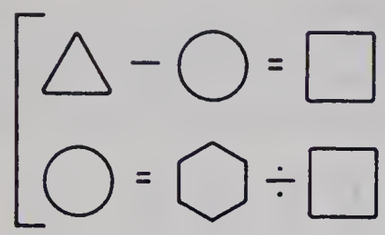


In example I, we are told that 3 and 1 are the pair. Then the sum, or 4, is recorded under the triangle; and the product, or 3, is recorded under the next hexagon. In example II, we are given one member of a pair (4) and the product (16). Since $4 \times 4 = 16$, the missing member of the pair must be 4; then the sum must be 8. In example III, the pair is obviously 5, 3 and the missing product is 15. In example IV, the product is 5 and one member of the pair, taken as a factor, is 5; therefore, the other member of the pair must be 1; and the sum must be 6.

An advantage of this presentation is that the questions may be asked in a variety of ways:



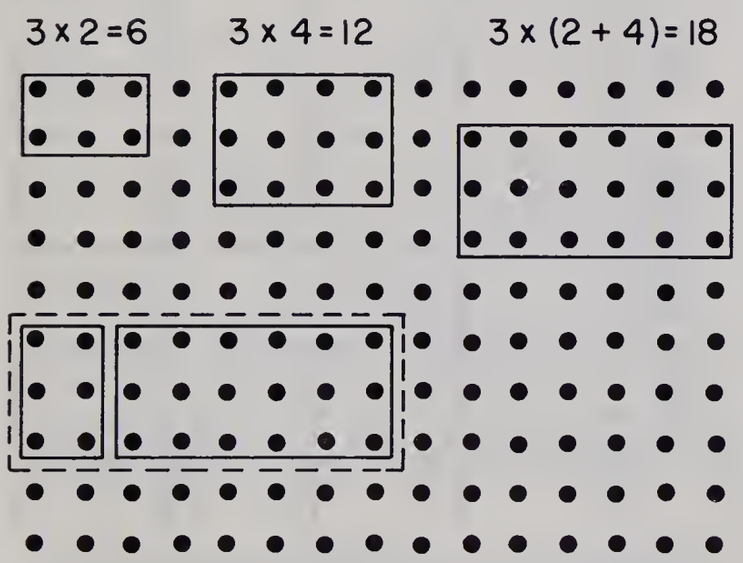
And, later, the same forms can be used in subtraction and division:



3. Pictures relating addition and multiplication

Begin with a large array of dots on the blackboard. Ask such questions as the following:

"Can someone draw a line around a group of dots to suggest 3×2?"
"Can someone draw a line around a group of dots to suggest 3×4?"
"Can someone draw a line around a group of dots to suggest 3 × (2 + 4)?"



After several similar examples, put the following three statements on the board:

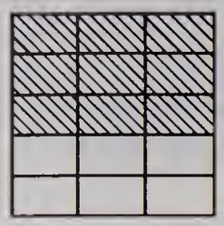
$2 \times 3 =$ ____
$6 \times 3 =$ ____
$(2 + 6) \times 3 =$ ____

"We can use one group of dots to suggest these three sentences. (On the board, mark off the section above that has been outlined with a broken line.) The section I have outlined on the board clearly suggests every part of the long mathematical sentence." Now, let's complete the sentences:

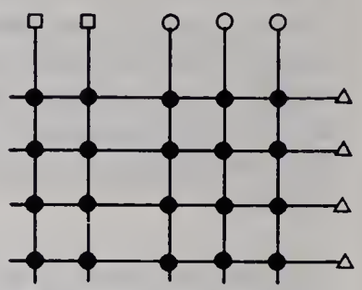
$2 \times 3 = 6$
$6 \times 3 = 18$
$(2+6) \times 3 = 2 \times 3 + 6 \times 3 = 6 + 18 = 24$

Next, draw several diagrams on the board and ask the children what mathematical sentences each suggests.

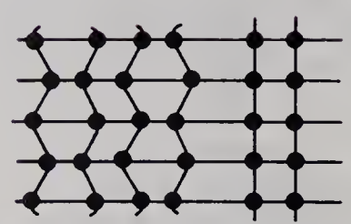
II



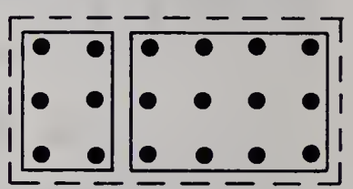
I



VI



III



Mathematical sentences suggested by diagram I would include:

$4 \times 5 =$ ____	$12 + 8 =$ ____
$5 \times 4 =$ ____	$8 + 12 =$ ____
$4 \times 3 + 4 \times 2 =$ ____	$20 - 8 =$ ____
$20 \div 3 =$ ____	$20 - 12 =$ ____
$4 = 20 \div$ ____	$(12+8) \div 4 =$ ____

Similar sentences are suggested by each of the other diagrams. After sufficient sentences for each diagram have been noted, reverse the procedure by asking pupils to find the diagram that is suggested by a mathematical statement. "Which of the diagrams (shown above) is suggested by:

$3 \times 5 =$ ____	$5 =$ ____ $\div 6$
$18 \div 3 =$ ____	____ $- 6 = 12$
$2 \times 3 + 3 \times 3 =$ ____	$10 + 20 =$ ____
____ $- 3 \times 4 = 2 \times 3$	$5 \times 6 = 5 \times (2 +$ ____ $)$
$6 \times 3 = 6 +$ ____	$20 - 3 \times$ ____ $= 2 \times 4$

Have pupils volunteer to make up sentences suggested by the diagrams. The rest of the class must tell which diagram is being referred to in each sentence.

4. Games of logic

These games can be played many times with the children. With them, they learn to reason mathe-

matically, and then to extend their ability at logic to other areas. Each "If..., then...," statement employs mathematical logic or reasoning. Many of these games are also exercises in precise expression. The games are not fun unless we are sure that everyone says exactly what he means. This gives you, the teacher, an opportunity to emphasize the need, in certain circumstances, for precise expression of ideas.

a. *What's My Rule?* In this game of logic, the leader first asks for a number to which he then applies a rule, such as *add 2; subtract 3;* or *multiply by 10.* Others submit numbers to which the leader applies his rule until someone thinks he knows the rule. He must prove his discovery by giving the second number when another number is submitted. Examples:

Numbers submitted by others–	1	3	10	8
Response of leader–	2	4	11	9

(Rule: *The next number, or one more.)*

Numbers submitted by others–	2	7	5	12
Response of leader–	4	14	10	24

(Rule: *Twice the number.)*

Below are listed some of the rules which might be applied to the numbers 2 and 4 in order to get a third number:

2,4	2	(the *smaller,* or *first,* of the pair)
2,4	6	(their sum)
2,4	8	(their product)
2,4	3	(their average)
2,4	4	(the larger, or second, number)
2,4	5	(the second number and 1)
2,4	24	(the numbers side by side)
2,4	12	(twice their sum)

It is also possible to move out of arithmetic with this form of "What's My Rule?" Put the following on the board. Can the children finish the examples and make up some of their own that follow the same pattern?

1) a→b
b→c
m→n
s→__
__→s
__→r
__→__
__→__

2) a→c
b→d
f→h
r→__
__→x
__→f
__→__
__→__

3) a→A
b→B
c→C
m→__
__→X
__→Y
__→__
__→__

4) c→b
e→d
t→s
d→__
m→__
__→h
__→i
__→__
__→__

5) c→a
h→f
n→l
r→__
q→__
__→__
__→__
__→__
__→__

6) a→B
f→G
r→S
b→__
d→__
__→T
__→Q
__→__
__→__

7) band→and
bone→one
bin→in
bark→_____
_____→owl

8) 23→3
37→7
10→0
99→__
__→8

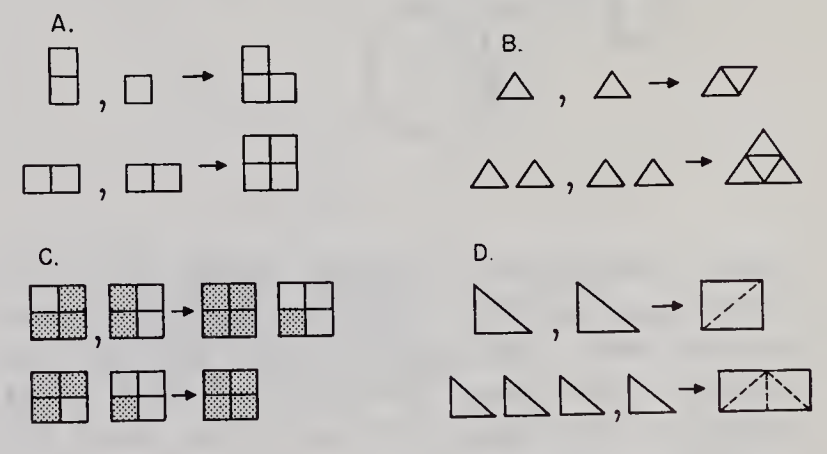
9) man→men
bird→birds
woman→women
mouse→_____
_____→feet

10) (present__past)
give→gave
have→had
hit→hit
run→_____
_____→fought
_____→_____

11) (reverse first and last letters)
now→won
saw→was
rat→tar
tip→____
pet→__
____→____

12) (reverse order)
123→321
37→ 73
8→ 8
25→__
____→34
____→____
____→____

Below are various examples that will help pupils see the relationship between the sizes and shapes of things:



E.
be, at→beat
my, self→myself
of, ten→often

F.
upon, on→up
into, in→to
today, to→day

G.
bother, am→wing
five, two→a
mother, up→year

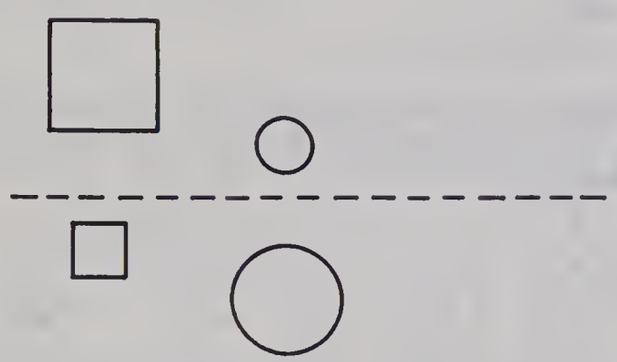
H.
mask, m→ask
tear, r→tea
swing, w→sing

In each of the above cases, a rule or a pattern is suggested by a few examples. As soon as pupils recognize the rule, they can complete other examples according to the rule, and they can make up examples of their own. There is no need to verbalize the rule—simply continue to illustrate it in examples.

The first child to understand the rule can be made an "expert" to test others who think they have discovered it. When two or three children have become experts, the teacher can help those who are still not tuned in, while the experts continue to test others.

Let children make up their own series to suggest a rule they have in mind. They can put their examples on the board, and act as "experts" for those who think they have discovered the secret. In this way, many activities can be going on at once without a teacher's direction, and the teacher can see that everyone succeeds in discovering some patterns.

b. *Fewest Questions.* Here is a game that strengthens number concepts, expands vocabulary and develops powers of reasoning. It is called "Fewest Questions." You can start with a very simple example on the blackboard or on a piece of oak tag.



Some preliminary questions might be: "How many shapes are there above the line? How many are there below the line? How many are there altogether? How many are squares? How many are circles? How many are large? How many are small?" Now to the game.

"I'm thinking of one of those shapes. Try to guess which one it is. We'll keep score of how many questions you ask."

Select a child as scorekeeper at the board to make a tally each time the guesser asks a question. Except at the outset, discourage having to point to the diagram in asking questions.

"Is it the big square?"

"No, it's not the big square. That's your first question."

"Is it the big circle?"

"No, it's not the big circle. That's your second question."

"Is it the little circle?"

"No, it's not the little circle. That's your third question. Do you know the shape I'm thinking of?"

Of course, if the guesser remembers the questions, he would know that it has to be the "little square" but he may keep on asking until he gets the answer.

After a few such guessing sequences (in which you may take the liberty of saying *No* as often as you logically can), offer to be the *guesser* yourself.

"I'm going to turn my back. Barney will come up to the board and point to one of the shapes. You will all know the one he selects—but I won't. Then I'm going to find out by asking just two questions!"

Your questions may be:
1) Is it a square?
2) Is it above the line?

There are only four possible combinations of answers. They are

	A	B	C	D
Is it a square?	Y	Y	N	N
Is it above the line?	Y	N	Y	N

If the answers to both your questions are "Yes" (A)—it is the large square.

If the answers are "Yes" and "No" (B)—it is the little square.

If the answers are "No" and "Yes" (C)—it is the small circle.

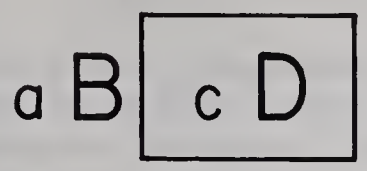
If the answers are both "No" (D)—it is the large circle.

You might ask another pair of questions.
1) Is is small?
2) Is it a circle?

Again, any combination of Yes-No answers will reveal the chosen shape. The secret is that each question must divide the group in half in a different way.

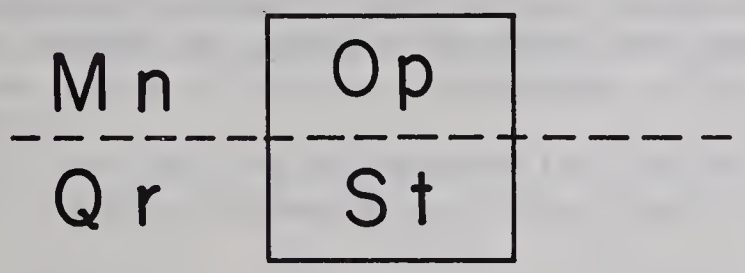
After this, you should return to the children

as *guessers* and *selectors*. (Some will get the idea right away. They've discovered the secret and they must keep it.) Change the diagram.



1) Is it a large letter?
2) Is it in the box?

After the children become adept at asking the questions so they never need more than two, enlarge the diagram.



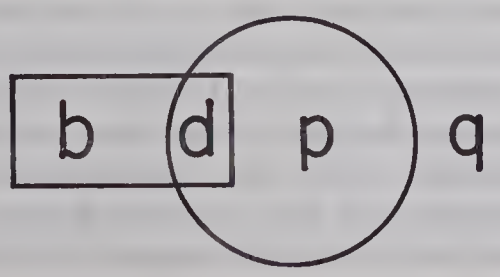
"How many large letters? How many small letters? How many letters altogether? How many letters above the line, how many letters below the line? How many inside the box? How many letters are both above the line and in the box? (Only two—O and P.) How many large letters are both below the line and not in the box? (Only Q.)"

That last question suggests a group of three questions that will reveal any choice:

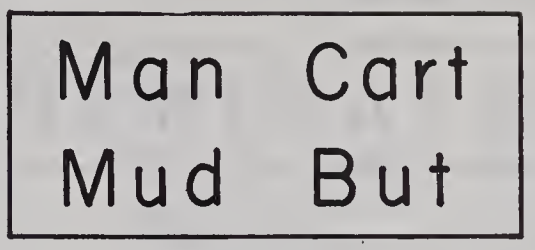
1) Is it a large letter?
2) Is it above the line?
3) Is it in the box?

(1)	Y	N	Y	N	Y	N	Y	N
(2)	Y	Y	N	N	Y	Y	N	N
(3)	Y	Y	Y	Y	N	N	N	N
	O	p	S	t	M	n	Q	r

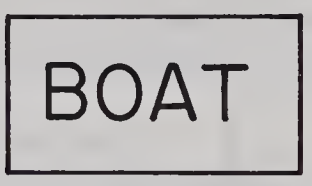
"Fewest Questions" is a favorite with the children. Let them take their time in discovering the secret. You may vary the objects to include geometric shapes, letters of the alphabet, short words, numberals, etc. Below are a few variations in design with examples of the fewest questions that will expose any item chosen.



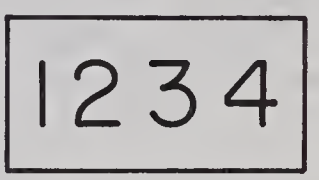
1) Is it in the rectangle?
2) Is it in the circle?



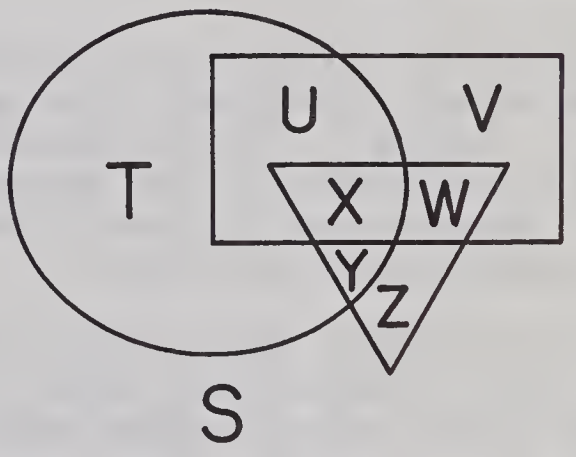
1) Does it begin with *M?*
2) Is there a *u* in the word?



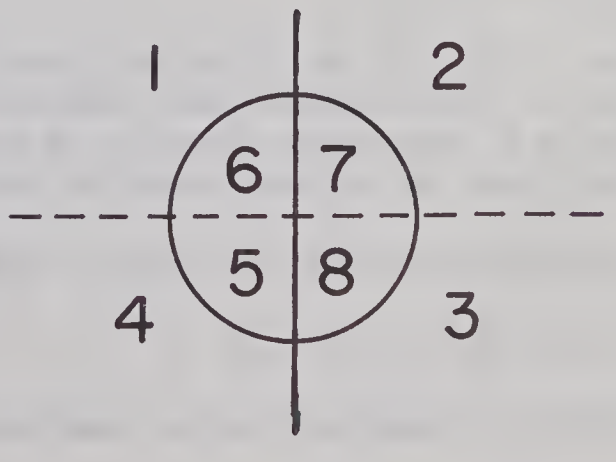
1) Is it in the word BARK?
2) Is it in the word TUB?



1) Is it one or four?
2) Is it more than two?



1) Is it in the circle?
2) Is it in the square?
3) Is it in the triangle?



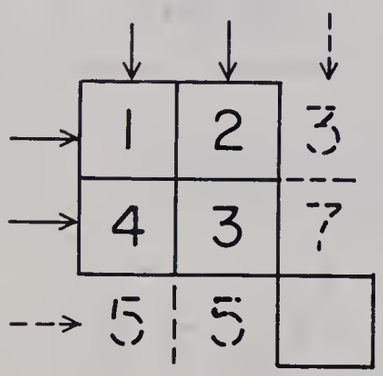
1) Is it above the line of dashes.
2) Is it to the right of the solid straight line?
3) Is it in the circle?

E T A O I N S H

1) Is it in the word WINTER?
2) Is it in the word FINISH?
3) Is it in the word EASIER?

5. The Cross-Number Puzzle

On the board, draw the following:



The arrows, you will explain, tell us to *add*. As results of the four examples of addition indicated by the solid arrows are recorded, we are led to wonder where to record the additions indicated by the broken-line arrows.

Of course, in one example shown, we could write 10 in the box and it would serve jointly as the result of 3 + 7 and of 5 + 5. Is this a coincidence? Most children and adults think it is, and that this outcome is dependent upon the choice of the four numbers originally written in the box.

Many of your children would be less surprised by the outcome if they imagined objects in the four cells—one dot, two squares, three circles, and four cars. We can sweep the objects across into piles of three and seven; then sweep them down into a single pile of ten. Or, we could sweep them down into piles of five and five; then sweep them across into a single pile of ten.

If we started with ten objects and swept them together we would not expect any to appear out of thin air or to miraculously disappear. If we start with ten objects, we will end up with ten objects.

We might keep track of the goings-on this way:

$$(1+2)+(4+3)=3+7=10$$

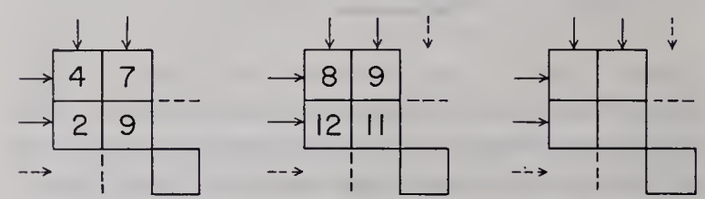
If we had, however, swept down both columns and then across the bottom row, we might keep the following record.

$$(1+4)+(2+3)=5+5=10$$

There is a fundamental rule in arithmetic that the order in which numbers are added does not affect the result. In columns of addition, we often "add up" the columns and then, to check our computations, we "add down" the columns, because we are sure that the order in which we add ought not to change the result.

Yet, when most children and most adults are first confronted with a Cross-Number Puzzle, they are amazed that the box in the lower right-hand corner will suffice for recording the addition in both the right-hand column and the bottom row.

Try the next two examples and then write any four numbers that come to mind in the third:

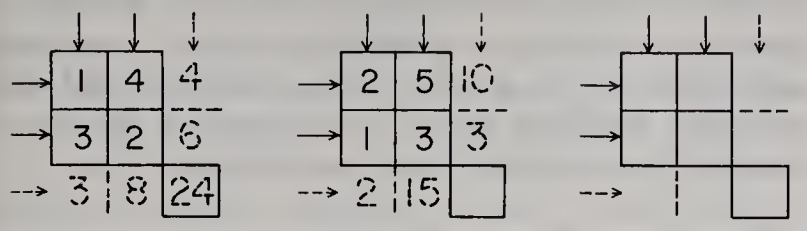


You probably have several children in your class who can give this puzzle a real work-out while trying to discover a group of four numbers that won't work. Even though they have swept objects on the floor and visually swept designs on paper, first one way and then the other, they do not fully comprehend that those numbers in the cells are answers to "How many?" How many what? Anything you like or don't like, or that can be counted.

a. In-any-order rule. In this work, you are helping children see examples of a basic idea in mathematics: *When we add, the order in which we proceed is immaterial.* We have nicknamed this rule the *in-any-order* rule. (Mathematicians will analyze it more carefully and tell us we are combining two fundamental laws, the Associative law for addition and the Commutative law for addition. The *in-any-order* rule for addition seems more descriptive and is easier to remember.)

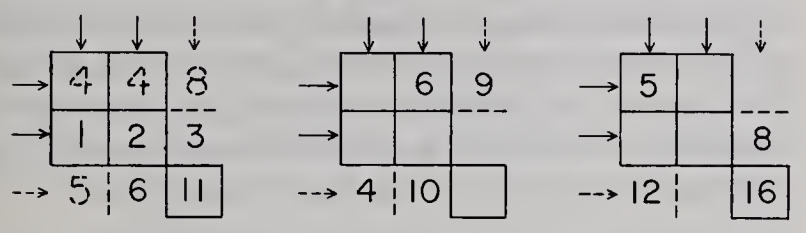
b. In-any-order works in multiplication. This same puzzle helps reveal that there is an *in-any-order* rule for multiplication, too. In the examples below, the arrows tell us to *multiply*. (In the third example, try any four numbers that come to mind.) Children will be helped by this puzzle form to learn what mathematicians call the Commutative law for multiplication and the Associated law for multiplication which we, for the sake of clarity, brevity

and usefulness, combine into the *in-any-order* law for multiplication.



c. Subtraction—the inverse of addition. The Cross-Number Puzzle will also be used to present subtraction as the twin of addition, its opposite or *inverse.* The agreement for the puzzle below is, as earlier, that the arrows indicate addition and that the bottom right total is the result of adding the four numbers in the basic four cells. We are now faced with a problem of reconstruction, or building backwards.

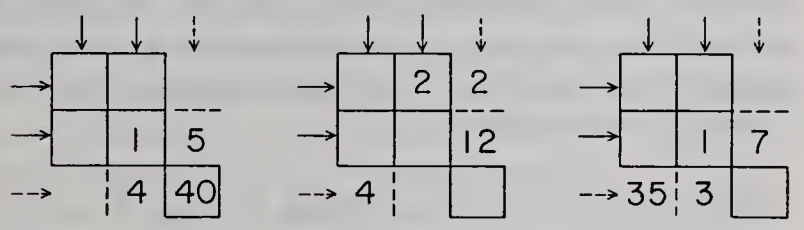
Addition:



To rebuild, in the first example, you may argue: I'm told that ____ + 3 = 11 or in other words 11 − 3 = ____. I know that 8 + 3 = 11 − 3 = 8. Therefore, "8" must go in the top cell on the right column of the first example.

d. Division—the inverse of multiplication. Similarly, we can reconstruct puzzles when the rule is that the arrows indicate multiplication.

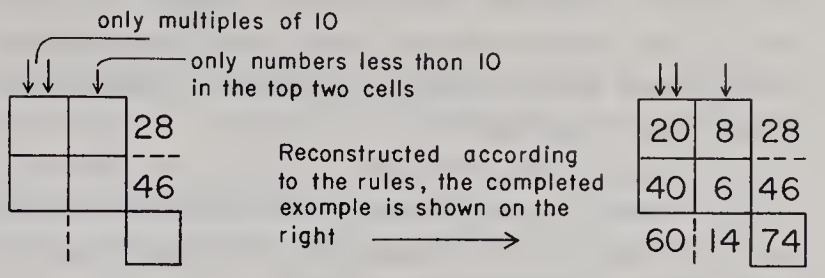
Multiplication:



In the first example the question is asked, "What number multiplied by 5 results in 40?" Or, "40 divided by 5 equals what number?" Etc.

The problem of reconstruction now involves *division.* Again, we have a dramatic example of division as the opposite of multiplication.

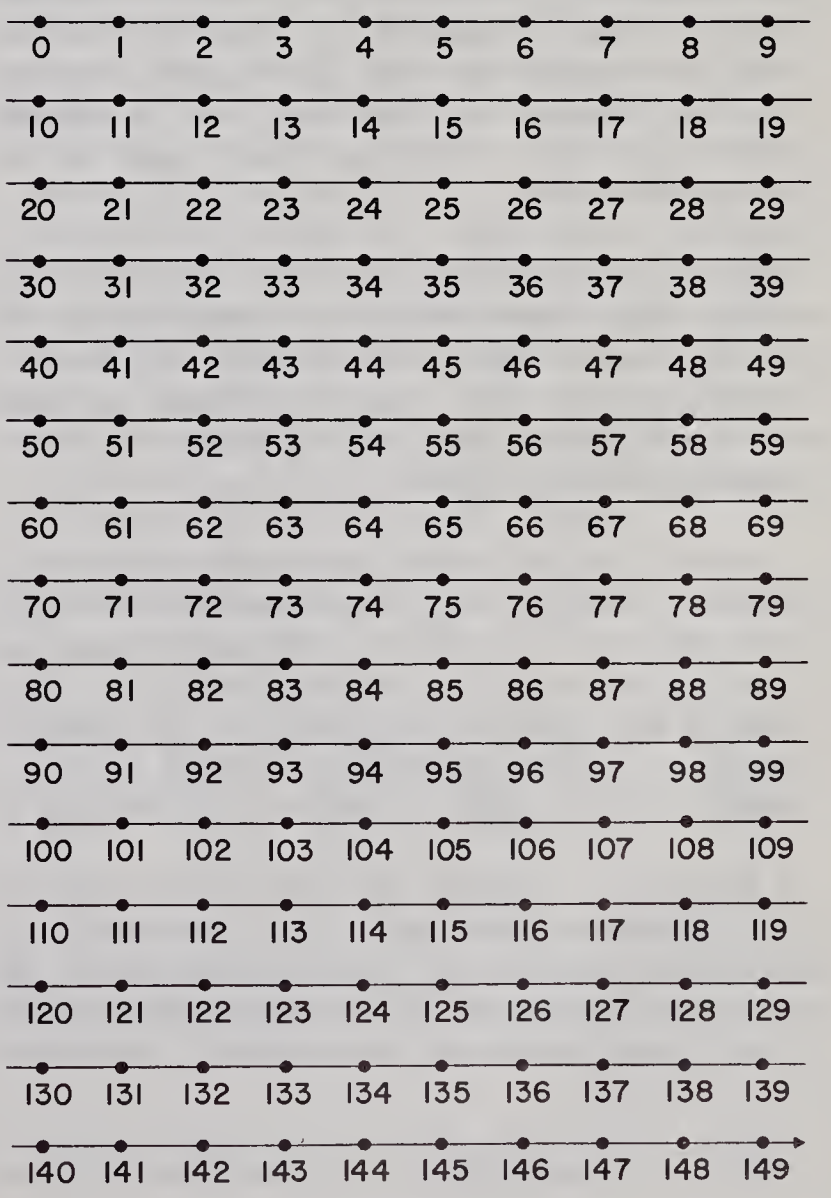
e. Help with algorisims. Cross-Number puzzles help with the algorisms children will later learn to employ when necessary. The examples below call for reconstruction according to the agreement that, in the column under the double arrow, only multiples of 10 can be used and in the column under the single arrow, only numbers less than 10 can be used in the top two cells. Otherwise, the normal rules of addition are used.


only multiples of 10
only numbers less thon 10 in the top two cells
28
46
Reconstructed according to the rules, the completed exomple is shown on the right
20 8 28
40 6 46
60 14 74

In following the rules, 28 and 46 are "regrouped" and then added in this regrouped form. This series of steps will be condensed or abbreviated and grow into the algorism of addition which we know.

6. Number-Line games

A large copy of a number-line chart like that shown below can be put to good use in the classroom. It can be used to demonstrate that it is possible to jump from 2 to 12, or 5 to 15, or 7 to 37 with great ease. Pupils won't need to count in order to do this, either. Every jump down "↓" is equivalent to adding "10." A jump down "↓" and ↷2↷ is equivalent to adding "12." A jump down "↓" and ↶2↷ is equivalent to adding "8." To find the sum of 10 and 17, you start at 10, jump ↓ one row and ↷7↷ and, magically, you're on 17. Or, start at 10, jump ↷7↷, and then jump ↓ one row.



Allow your pupils to spend several days in learning how to move about the number-line grid. They will enjoy exercises like these: "Where will you land if you start at 0 and take three jumps down? Four jumps down? Two jumps down? Six jumps down? Ten jumps down?

"How many ways can we make the trip from 11 to 22? From 5 to 12? From 3 to 24? From 7 to 18? From 25 to 37? From 111 to 123?

"Where are we if we start on 15, jump down two rows and four jumps to the right? (15 + 20 + 4 = 39) If we start at 15, take four jumps to the right, and jump down two rows? (15 + 4 + 20 = 39) If we take two jumps to the right, jump down two rows, and jump two to the right? (15 + 2 + 20 + 2 = 39)"

Have pupils jump forward or back, and then up, to subtract on the number line. "Just as there are several ways to get from 15 to 39, there are also several ways to get from 39 to 15."

When pupils understand how to use the number-line chart, it is possible to play games with it. For instance, one game is started by saying, "I'm thinking of a number. Start at 15, jump three rows down and four to the left. What is the number?" Whoever can tell you first what the number is becomes the leader and has to think of a number for which he must give appropriate clues. Some pupils may become so proficient that they won't need to use the number-line chart. Encourage such pupils to "do it in their heads."

Other games could proceed like this: "Start at 0, add 11. Add 11 more. Add 11 more. Where are you?" Or, "Start at 3, add 6. Add 3 more. Add 6 more. Where are you?" Use repeated additions and subtractions of 3,4,5,6,7, etc, and ask pupils to look for patterns as they progress.

Jumping on the number line can be used to help underline the fact that in subtraction we simply *undo* what we *do* in addition. If *jump down one row* is equivalent to adding 10, then *jump up one row* must be equivalent to *subtracting 10.* If a jump of *two to the right* is equivalent to *adding* 2, then a jump of *two to the left* is equivalent to *subtracting* 2.

a. Regrouping numbers. In order for a pupil to add 8, it is possible to add 10 and subtract 2, or to subtract 2 and add 10. Confronted with 17 + 8, pupils can find the sum by regrouping the 8 in such a way that the sentence becomes: 17 + 10 − 2, or 17 − 2 + 10.

It is not possible to outline a set of rules for regrouping numbers. As each pupil gains confidence in putting the idea to use, he will also gain an appreciation of when regrouping is especially helpful to him in simplifying a problem. As they jump back and forth on the number line chunks to add and subtract, children learn to feel what's going on, because they *see* what they are doing.

To subtract 3 from 61 on the continuous number line is simply a matter of taking a jump of 3 backwards, but on the number-line chart 61−3 must be handled differently. The problem has to be approached as 61−10 + 7, or 61 + 7 − 10.

b. The decimal system and shortcuts. The structure of the decimal system is emphasized in the way the number-line chart form is designed. Bases for understanding the algorisms of adding and subtracting can be firmly established by developing the ability to jump freely on the chart. Enterprising children will discover many shortcuts for making jumps. The search for shortcuts should be encouraged, because in discovering shortcuts, pupils also discover basic patterns in the decimal system.

7. Discovering matching properties

Research on *discovery in arithmetic* has revealed important information. When children discover a relationship, they are excited and anxious to stay with that discovery. They will eagerly put this new idea to work, showing more examples of *it.* They come alive, often with an "Oo-o-o! I've got it!" or with a burst of industry using a newly discovered relationship to reel off answers with confidence.

What is the *it* they have glimpsed? Well, if we distract their attention with a request for them to *tell about it* or *put it in words,* we have selected a method that could not be better calculated to mutilate the experience.

That a child can move ahead, putting *it* to work successfully is certainly all the evidence we need to know he *has it.* Our only justification for veering off in the direction of a semantic exercise is a feeling that *it* is a generalization that ought to be verbalized. But research does not support this as a valid reason.

Transfer to future experiences is not increased by *verbalizing a generalization.* Sub-vocal awareness, the new-found ability to move ahead more speedily, more directly, more confidently, the discovery of a new and important relationship, will stay around as an effective tool even though it hasn't been named and talked about. An opportunity to put this tool to work will contribute most to its staying power.

8. Silent teaching and discovery

The silent technique of instruction has revealed the mechanics of discovery unencumbered by any attempt at verbalization. A class can be kept on the edge of their seats for an uncommonly long time (and, incidentally, almost alarmingly quiet) as you move on from one level of difficulty and involvment to another. Though the way leads from the familiar deep into new, unfamiliar territory, interest is high, reactions quick, and with the teacher's eyes moving to the fingers, ordinarily timid children can be drawn into participation. The most retiring or placid child may, all at once, seem to come alive. Children who usually are ready first with the answers, are not as effective in keeping in the center of the spotlight.

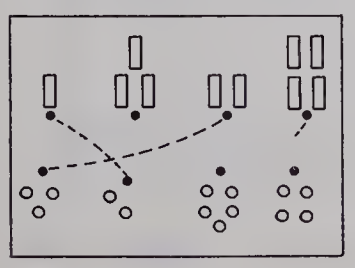
At the conclusion of a silent session, there is really nothing to talk about. And no one seems interested in talking about the experience. But as soon as you lose your voice again, there is an immediate awakening of the feeling of excitement. If a similar activity is initiated several days later, it will move faster than before. No generalizations were verbalized, yet something of what happened has been transferred from the trips into silence.

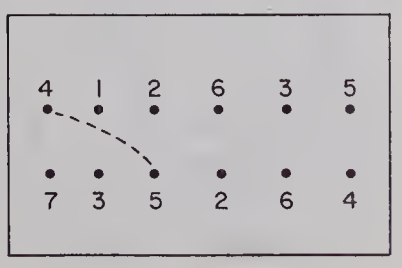
The only danger involved with arithmetic without words is that of over-stimulation. *Discovery* is not a relaxing state, physically or mentally. An act or moment of discovery is a moment of elation that seems to release a dynamics that can be almost self-consuming. The look of eagerness on so many faces all at once is infectious and it is tempting to see how fast and how far you can go.

The best way to assure a moderate forward movement is to concentrate on participation by everyone. If it's a board activity, everyone should handle the chalk. If it's a manipulation of blocks on a table, everyone can show he knows the secret.

One rule of teaching that will multiply the opportunities for discovery in any activity, arithmetic included, is to use the fewest possible words. And when you must talk, leading questions are much more fraught with educative value than any explanations you can make.

a. The game of "Next After." Without any words, put the following examples on the board:





The rule you have in mind is, of course: *One more,* or *the next label on the number line in a direction away from zero.*

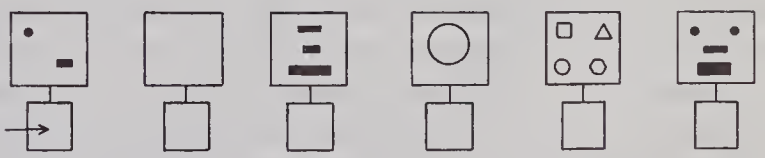
Complete the first connecting line, one square to two circles. Begin the next line and stop, looking for volunteers. Many children may think you have made a mistake. You must show confidence in your connecting. Some volunteers may make unsuccessful guesses as to your secret rule. If no one guesses it, you may have to draw the connection between the dots associated with two squares and three circles.

Start a third connecting line from the dot under the four squares. At worst, a child will connect it to four circles. You are troubled. You retrace the first and second connecting lines. With the chalk, you touch the single square and then the two circles. You express approval. Then you touch the two squares and the three circles. Again you are pleased. And again you look for volunteers. You may even have to complete the entire first example by yourself, though this would be unusual.

Move on to the second example. If no one has read your mind in the first example, begin by connecting the dot associated with four in the top group and five in the bottom group. Proceed as in the first example. You will be surprised by the reaction of the children. This time they will get the idea.

Why? Experiments lead to the conclusion that children understand the "Next After" *numerical game* quickly when it is associated with a memorized sequence of words or symbols rather than in the context of a group that has one more member than another. If you encounter this development, place increased emphasis in counting collections of things and arranging them in order from smallest to largest and from largest to smallest.

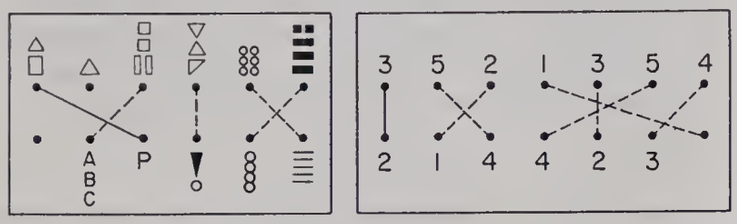
Erase the two examples and go back to labeling pictures of groups.



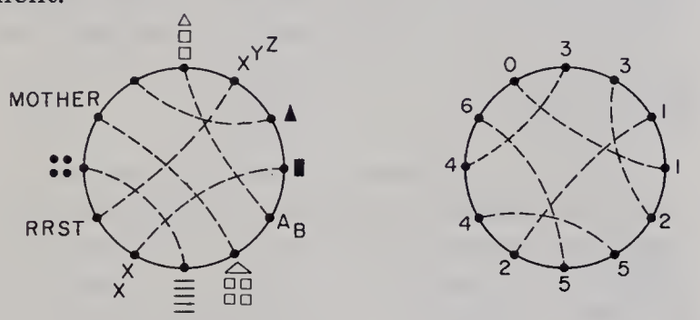
Then go on to another series of pictures. Skip the first, put one design in the second, two in the third, and stop just as you are about to start the fourth. Any volunteers, you seem to ask?

The idea is, of course, the same as with the original pair of examples: *one more,* in another context.

b. More silent teaching. If you feel that your class's response is advanced enough, go on to more advanced silent treatment examples. Use pairs of examples, one with designs, the other numerals.

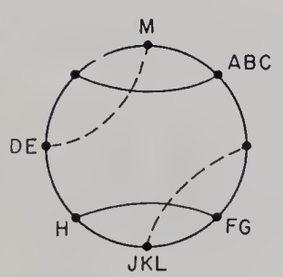


Starting at a dot associated with the pictures in the top row, draw a line to a dot above a group with one less in the bottom row. (Note that *0* or *none* is *one less than one.*) Then move to a circular arrangement:

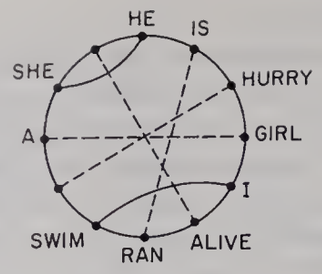


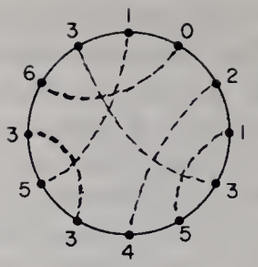
Now, one more and one less are two different ways of looking at a line. There is no general pattern of *from top to bottom* as in the first examples. Further, there are alternate ways of completing the connections. Both illustrations are quite similar. The second can be seen as recording the answers to *how many* designs at the corresponding point of the first. But, the connections are drawn to illustrate how different they may be in the two cases.

Jump to another rule. It also has a name, but there need be no hurry to label it.



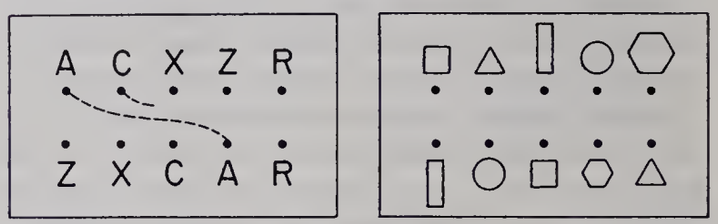
Make two connections, then look for volunteers. This is far from obvious, but it is one of the most important ideas in arithmetic. Our hope is that children will be able to read your mind and make right connections many times and in many contexts before we name the idea *addition.* Try an extension of the idea:





The secret in the first example is *five*; in the second, *six.* Some children may be thinking along these lines, but it would spoil the experience to talk about it. They are reading your mind. They glimpse a pattern that must parallel yours because they can help you draw the connections your patterns would dictate.

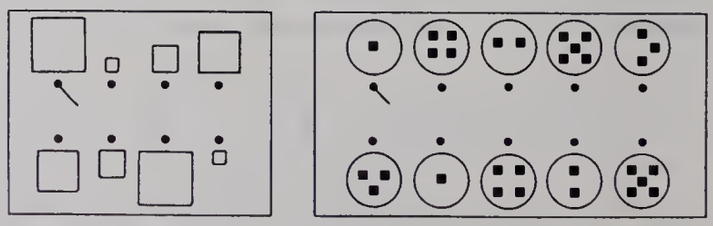
c. Matching designs. You have laryngitis again. Put examples such as the following on the board:



Start drawing a line from the dot under the "A" and stop. You don't have to do this job you were about to do. Look around for volunteers. There are probably several already. Hold out the chalk to one. If he continues the line you started with until it connects to the dot over "A" in the bottom row, you beam from ear to ear because your mental telepathy was successful.

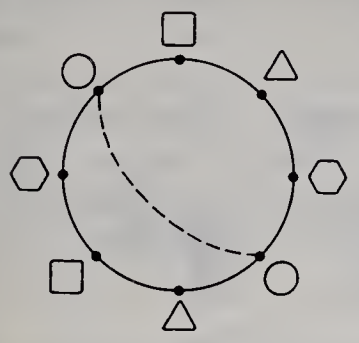
If the volunteer does anything else, you are upset because your telepathic effort wasn't well enough carried out. Remember, you have not said a word, no one knows the rules for sure, no one can make a mistake. You have to try again, and again until you have *communicated.*

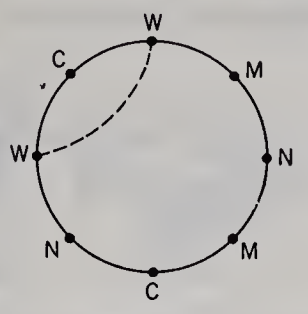
Start a line from the dot under the "C" in the top row. Again decide against extending that effort and look for volunteers. There will probably be many by now and you can move rapidly through both examples of matching. Next, try examples such as these:



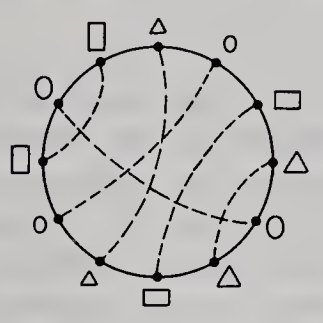
A similar activity, a bit more sophisticated,

develops when the objects or pictures to be matched are not displayed in two rows but, rather, around a circle.

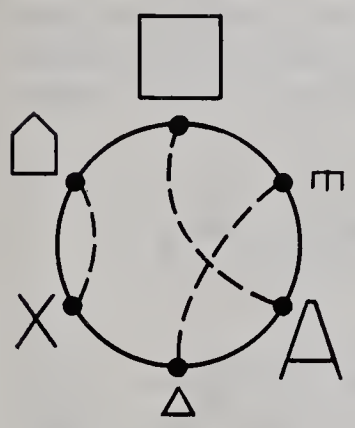


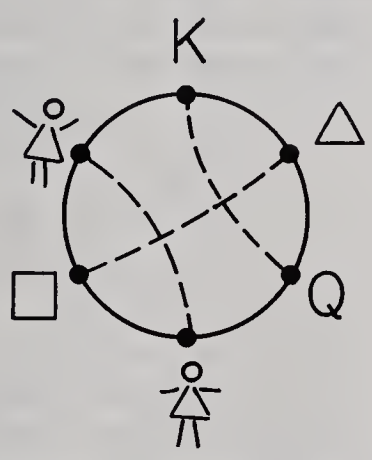


Then more matchings and more sophistication. (In the second example below, there are four of each kind of symbol, so the connections can be made in several ways, any one of which is as good as any other.)

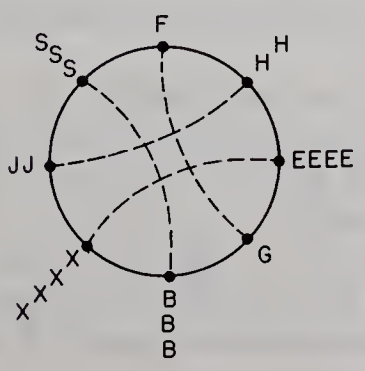


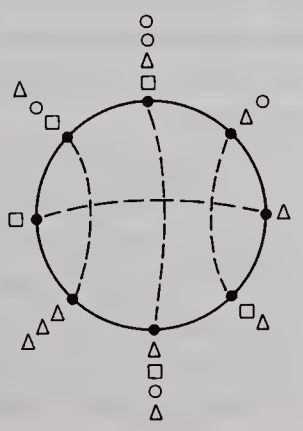
d. Largest, smallest, in-between. You still have acute laryngitis. In the first example below, largest, smallest and in-between, are the basis of matching. In the second, letters, shapes and people.



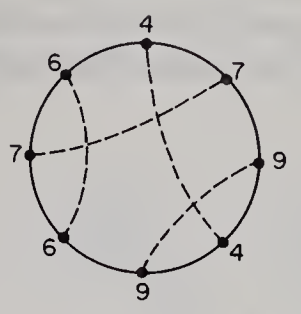


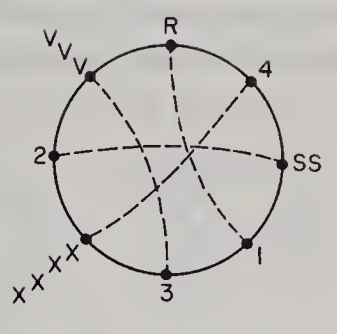
More examples:



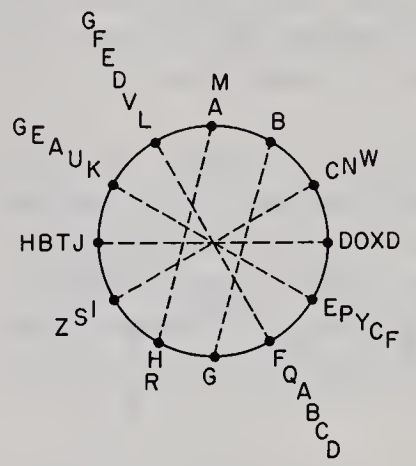


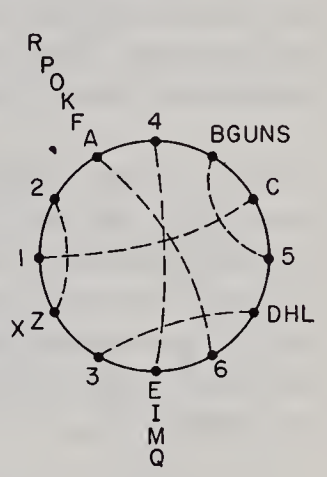
And on to the following:





You still haven't said a word, but activity is going ahead rapidly.





We have come a long way in a short time, moving through several very different kinds of matching, changing rules as we went, all without a word. This technique was first developed as a suggested answer to the question: "How could you teach arithmetic to young children if you did not speak nor understand their language?" Experiments led to the conclusion that you might find a real advantage in not being able to let the confusion of words distort the pictures.

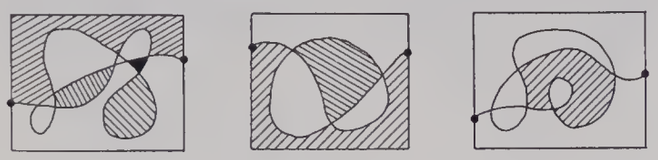
Number is an abstract idea. It is a *property* that groups of objects or other collections may have in common. If they do have this property in common, we say they "have the same number" and we assign a name and a symbol that serves to point to that particular *number* or *property that is common to a whole collection of groups.*

The sum total of the preceding paragraph illustrates what happens when we try to talk about the idea we were freely communicating to children before we resorted to *words.* Every matching was based on recognizing *common properties.* We ended with a matching between collections of letters and the symbol indicating their *number,* indicating the collection of collections to which they belong when number is the property we are noting.

9. The study of regions

Many experiences usually classified as art or "free expression" can be given a mathematical

orientation. For example, draw several rectangles on the board. Ask children to draw a continuous line from one side of the rectangle to the other, letting the line wander about the space. You may have results such as this:



Now, we must define a space or a region. Suppose the frame of the first rectangle is a fence and the lines we drew are fences, too. Imagine a cow eating grass. The cow is located at *a*. Color the area where there is grass the cow can eat.

In the next rectangle, imagine there are two cows. They are in different pastures. Where might they be? One might be at *b*. Color it in. (This area is chosen to demonstrate that the outside of the rectangle is also considered as a fence.) Where might the other cow be? Have children point to the area, and color it in.

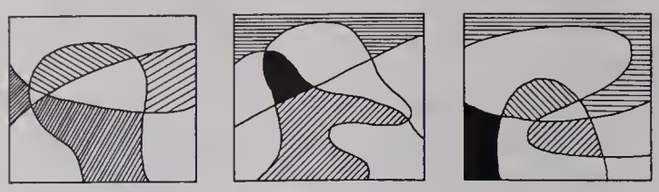
In the third rectangle, imagine there are four cows, each in a different pasture. Here are four different colored pieces of chalk. Use a different color for each pasture. (Four different children can work on this example, one after the other.)

Next, to individual work. Have everyone draw a wandering fence with a black crayon on a piece of paper. Have them draw another wandering fence in black crayon on the reverse side of their papers. On the first side of the paper, they are to imagine there are three cows, each in a different pasture. On the reverse side, they are to imagine there are four cows, each in a different pasture. They are then to color in the pastures where they imagine the cows to be.

The number of cows can, of course, be varied at will. After the activity has been tried once or twice, your directions may change. Two cows are in red pastures and three cows are in blue pastures. With a red and a blue crayon, the children should color in those pastures. Write a summary of instructions on the board:

2 red pastures
3 blue pastures

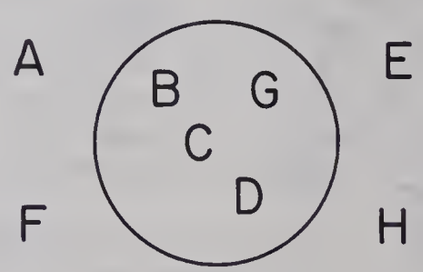
Here is another way to get a design started. Draw several rectangles on the board again. Have one child after another follow these directions. (1) Draw a line that starts at one side and goes across to the other but doesn't cross itself. (2) Draw another line that goes from one side to the other and crosses the first line. (3) Draw a loop that starts at the bottom, goes up and then comes back to the bottom (or from top back to the top). Examples might look like this:



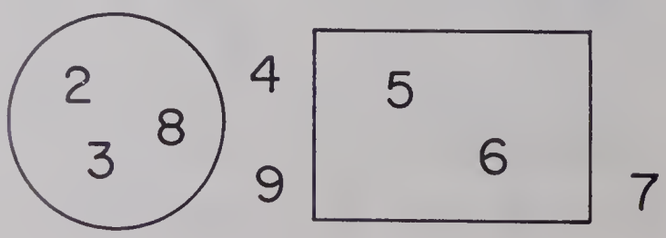
With different colored chalk, shade in any desired number of areas, regions or pastures. Now, as before, all the children can work independently. On one side of their papers, they might be asked to color three pastures or regions one color and three another color. On the other side, they might be asked to color six regions different colors, or all in the same color.

You may want to have the children write a number in the lower right-hand corner of their papers that tells how many spaces are colored—a title for the design. Drawing straight lines, curved lines, lines that cross, seeing intersecting lines create interesting regions are all ideas that will extend into the study of spaces that we call geometry. Following simple directions that lead to surprisingly different results each time, is a valuable experience.

a. Outside and inside. Explain to the class that a single closed curve of any kind establishes two *regions:* the *outside* and the *inside*. To demonstrate this, draw the following diagram on the board:

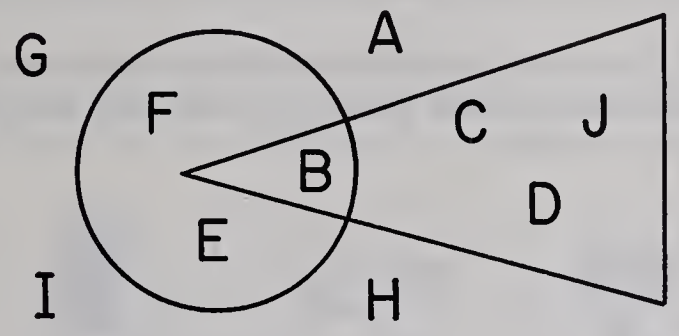


Ask, "Which letters are in the *inside region*?" Which letters are in the *outside region*?" Next, explain that the number of regions to be considered increases if more than one figure is involved. Draw the following diagram of a circle and a square on the board:



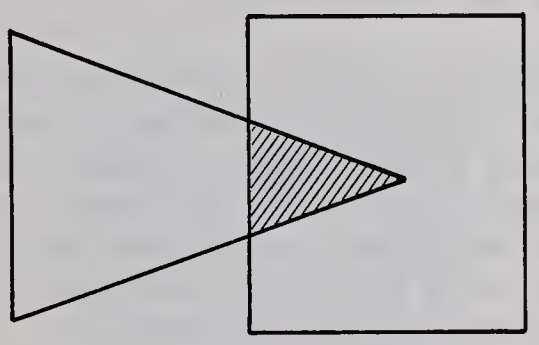
Ask, "How many numerals are in the region outside the circle? (5 and 6 are *inside the square, but they are also outside the circle.* The answer, therefore, is 6.) How many numerals are inside the square? (2) How many numerals are either inside the square or inside the circle? (5)"

Overlapping shapes make it especially necessary to be precise about what region is being discussed. Draw the diagram:



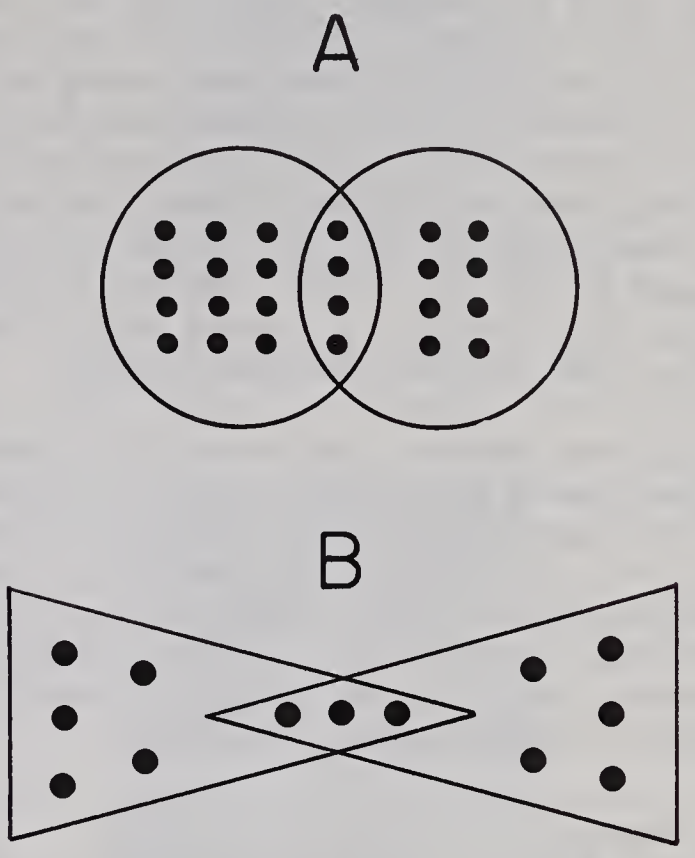
Ask, "How many letters are inside the circle? (3) How many letters are outside the circle? (7) How many letters are inside *both* the circle and the triangle? (1, "B") How many letters are *not* inside *both* the circle and the triangle? (9—all, except "B") How many letters are inside *either* the circle or the triangle? (6) How many letters are *outside* both the triangle and circle? (4) How many letters are *inside* the triangle but not inside the circle? (3)."

Draw the following diagram on the board exactly as it is shown here and have each pupil copy it:



Pupils should use a red crayon to color the region that is *both* inside the triangle and inside the rectangle. Then have them color the region that is *inside the rectangle but not inside the triangle* with a blue crayon. They should then color the region that is *outside* the rectangle with a green crayon. This latter region would include all of the triangle that does not overlap with the rectangle.

b. "I'm thinking of a region." Using a different colored chalk for each *region,* draw the diagram that is shown below on the board. Then, play the game "I'm thinking of a region," using this diagram as a model. Any region can be identified by the shape or shapes that form it and by the colored chalk used to delineate it. The object of the game is to guess the region the *leader* has in mind and to do so by asking a minimum of questions. Some pupils will discover the technique of cutting down the possibilities quickly, and will master the game easily.



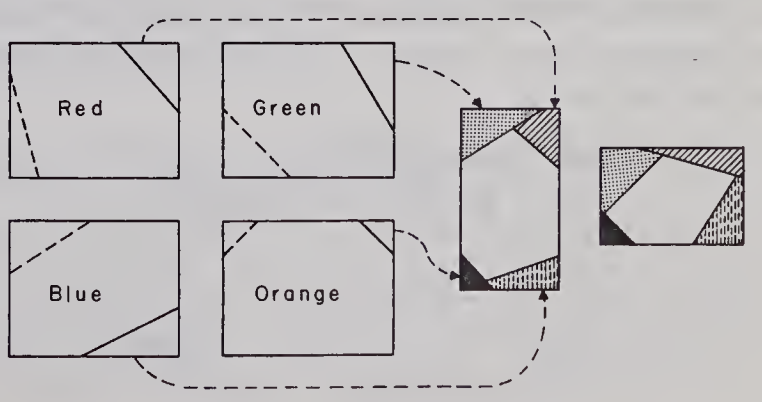
10. Making triangle designs

Suppose now for a moment that we had four pieces of colored paper—red, blue, green and orange. We could cut one corner off each sheet, some big pieces and some smaller pieces, and paste these four corner pieces on a new sheet, fitting the corners we cut off into the corners of the new sheet.

"Here is a picture of four sheets of colored paper. Who can draw a line on the sheet marked red to indicate a corner that is cut off?" Ask the same question for each sheet. Urge some children to make big pieces, small pieces, long thin pieces, etc.

"Now, let's look at the first picture frame. It's empty. What would it look like if we pasted that red corner on the picture, fitting it neatly in the corner?" Do the same with other four colors.

Repeat the activity, starting with a line that indicates another corner cut off the red piece. Construct a second design.



Let the children repeat the activity working either individually or in small groups.

Right triangles and similar triangles. The corners with which the children are working will later assume the dignified name of *right triangles.* The idea of *fitting* as it is used here becomes very important when we want to talk about *angles of the same size,* a condition which exists when one angle can be fitted into the other. When each of the angles of a triangle can be *fitted* into an angle of another triangle, we call them *similar.* The idea of angles fitting or not fitting is a required concept when talking about a shape or comparing shapes.

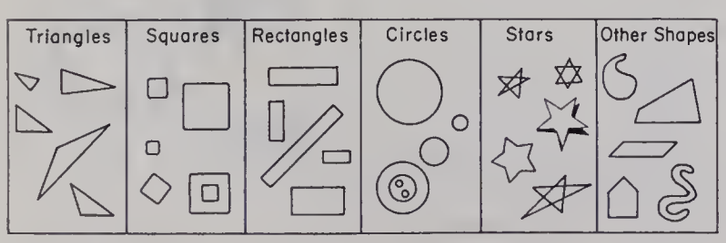
When the children have finished their corner-frames, there is a space left in each in which they can draw any pictures they like. Either at this time or later, other triangles, shapes with three sides, shapes with three corners or three points, can be cut from the remnants of the colored paper to make another kind of design. Put several examples of *triangle* designs on the board.

11. Drawing shapes

Section the board and write a name in each—triangles, circles, squares, rectangles, stars, other shapes.

"This section is for triangles. Who wants to draw a triangle in this section? Another triangle? A long thin triangle? A small triangle? A triangle with a very sharp point? A triangle with two very sharp, thin points?"

Then to *squares.* Bigger squares, smaller squares, a square inside of a square, etc. Whenever a volunteer draws a shape that doesn't belong, talk about it. "Does everyone think this shape is in the right place? Where should it go?" If it can't go in any other specific section, it can always go in the section titled "Other Shapes." No contribution need be excluded; some merely have to be redrawn in another section.



Now ask each child to draw at least one shape for every section—a triangle, square, etc. Then have them cut them out. Encourage the children to use different colored paper and to see if they can use up all of their scrap pieces.

Making designs. "Pick five of the shapes. Paste them on a sheet to make a design you like. Then take six more shapes and make another design. In the right-hand corner at the bottom, write the number that tells how many shapes there are in the picture."

On the board, sketch examples; or have some of the children sketch them. (Use several different colors of chalk.)

Have each child select a design he has made, the one he likes best. Line them up on the chalk rail or on a bulletin board. Give each a letter or number designation.

Pupils choose best design. The children are a jury. The job of each child is to study all the designs and pick the three he likes best. He is asked to write the letters or numbers of these three on a piece of paper and put it in a ballot box.

A committee is chosen to record the votes, giving one point to each picture named on each ballot. The tallying is best handled by a small group and the several designs with the highest votes are displayed prominently. (In this way, no one need be embarrassed by a public announcement that his design had none or very few votes.)

Should you discuss the three or four or five selected designs? Very briefly and not in detail, if at all. Children are good art critics but they are not able to express their basis for preferring one design to another. They respond to balance, symmetry, movement, color harmonies, etc., without having the background to verbalize such reactions.

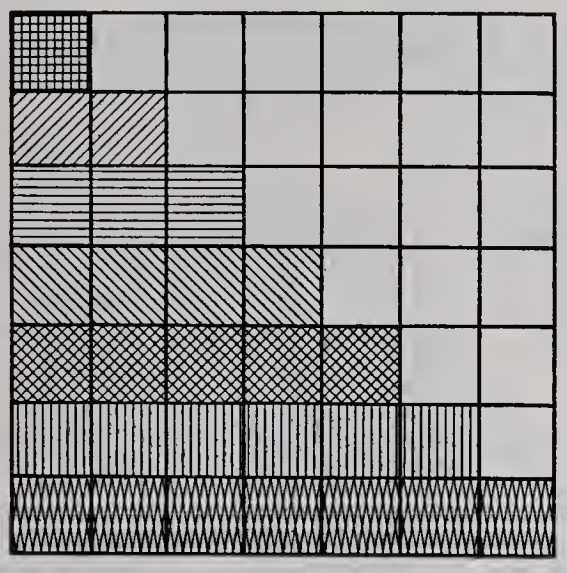
It is enough for the children to have participated in selecting favorites among a whole collection of abstract designs. There are no people or trees or cars that may or may not look like people or trees or cars to everyone. The children have made a judgment. *Likeness* was not the basis. What was the basis? You will never know, but it will be a mixture of balance, symmetry, movement, color

combinations and all the qualities that an art critic can verbalize.

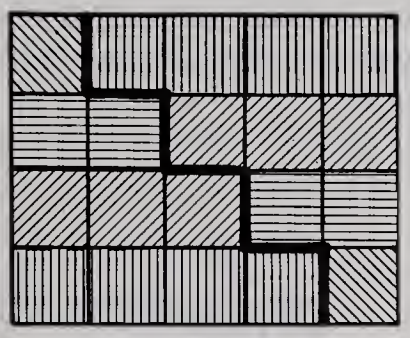
12. Graph-paper arithmetic patterns

Squared paper invites activity. It is never too early to begin; few persons will ever grow beyond graph-paper arithmetic – in mathematics, in engineering, in business, in reading the newspaper. No more effective methods have been discovered than work with graph paper to help reveal the simplest to the subtlest relationships.

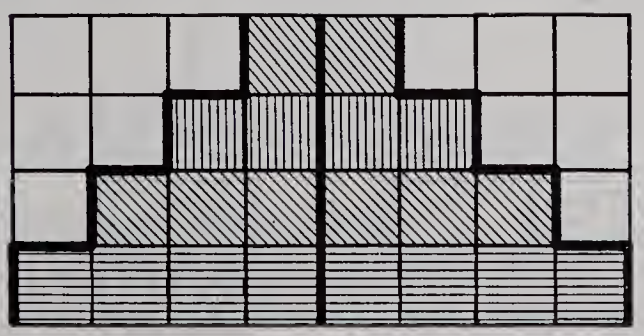
Stair-step patterns. Begin with the lowly *stair step.* Color a square with any crayon. Then the one under it and another one a different color, etc.



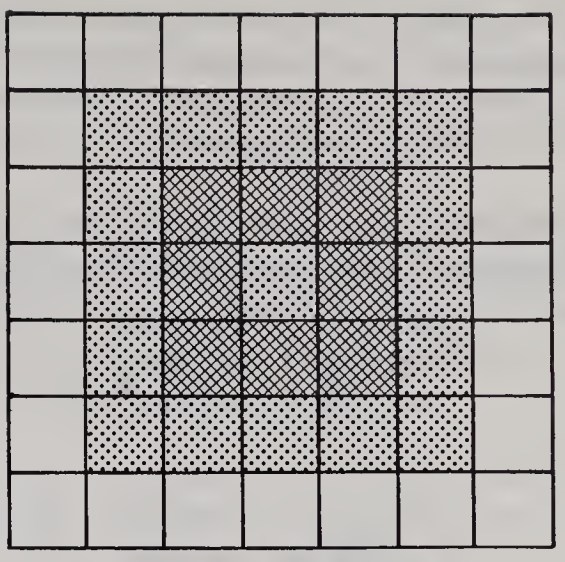
Make another set of *stair steps* up to ten high. Then, repeat the color combinations in an *upside down* set that fits on top.



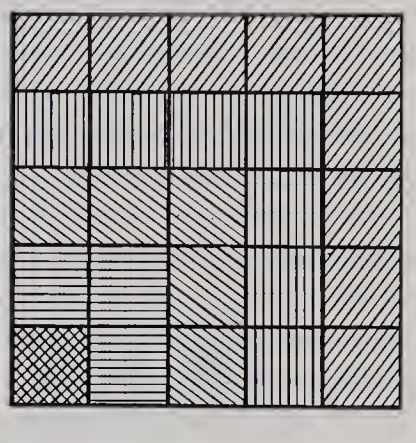
And, *back-to-back stairs.*



Nest of squares patterns. A *nest of squares* is an interesting pattern. Shade one square one color. With another crayon, color the squares around it.



Let a *nest of squares* grow out of the corner.

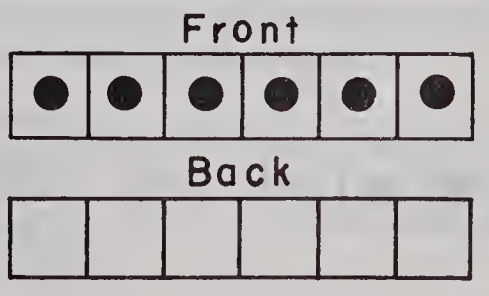


Later, this last pattern will help reveal the connection between *odd numbers* and *square numbers.* Each layer is composed of the next odd number of blocks – 1, 3, 5, 7, 9, etc. Each successive square is composed of the next square number – 1, 4, 9, 16, 25, etc. The pattern suggests the following table:

$1 = 1$
$4 = 1 + 3$
$9 = 1 + 3 + 5$
$16 = 1 + 3 + 5 + 7$
$25 = 1 + 3 + 5 + 7 + 9$

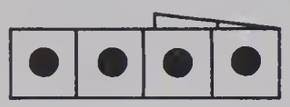
13. Counting activities

A most helpful teaching aid can be made easily out of paper, cardboard, or oilcloth. Establish a convenient unit, about three inches, more or less, and cut strips 2, 3, 4, 5, 6, 7, 8, 9 and 10 units long. Divide each into unit square boxes and put a large solid black circle in each unit square on one side. On the other side, mark off the unit squares, but leave them empty. The strip six units long will look like this:



Fold along each line that separates each strip into boxes so it can be easily folded along any divider. Show a strip to the class. "How many dots do you see?" Show them the back.

Now fold over 1, 2 or 3 dots. "How many do you see now? How many will you see if I turn it around?

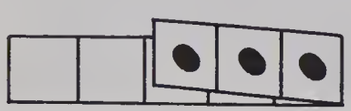


Turn it around so it looks like this:

"How many will you see if we straighten out the strip?" (Six.)

Take another strip; fold it before the children see it, and show them the side we will call the back.

"How many dots are on the side you can't see?"

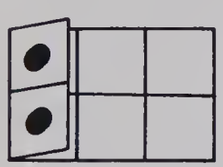


There will be five. "How many are there all together?" There will be eight. Straighten it out and count them. Then fold it differently:

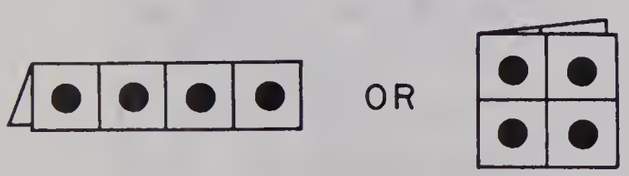
How many do you see? How many are on the other side? How many will you see if we straighten it out so you can see all the dots?"

Let different children fold the strips and display them, asking the questions. You may want to develop games with competing teams.

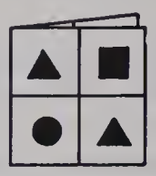
Make another set of strips, two dots deep. An example of such a piece folded might look like:



"How many dots do you see? How many are there on the other side? How many if you can see them all?" Or, it might have been folded thus:



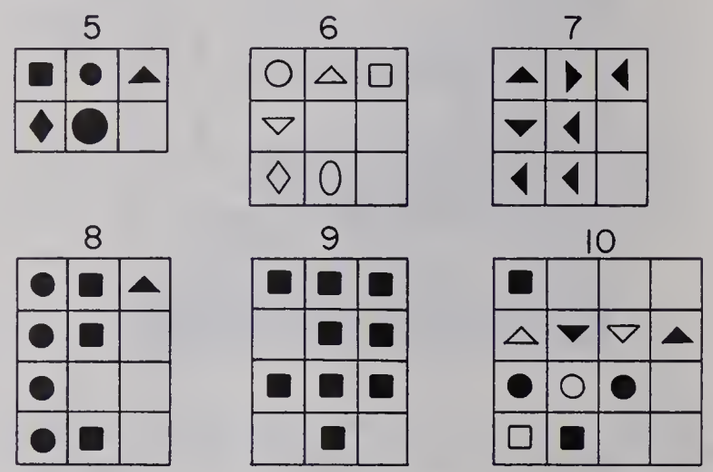
You may prefer using a variety of shapes in place of the dots:



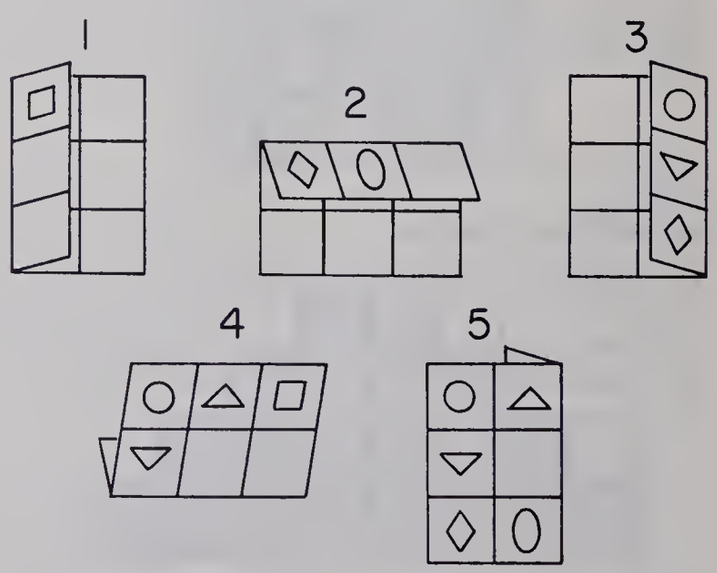
These two sets of strips and rectangular arrays involve an element of mystery, at first. "What's on the side you can't see? How many will there be when you can see them all?" As soon as the element of mystery wears off, many of the facts of arithmetic are ready in the children's minds for immediate recall.

14. Another aid for counting games

Some examples of another useful set of teaching aids are given below.



Each one of these is carefully designed so that it can be folded to display any number of the shapes desired. For example, the piece with six shapes can be folded as follows:



And, without folding, six shapes can be seen. Since each display can be folded to suggest every number from one to the total number of shapes, these few

pieces can be used to suggest every facet of addition and subtraction in which the addend is five through ten. The idea can be extended in both directions.

Show the unfolded card. "How many shapes do you see? Count them. Does everyone agree that there are nine?" Fold the card. "Now, how many do you see? Does everyone agree that we can see only four? How many will you see if I turn the folded card around? How many did you all see before I folded the card? Nine. And you see four now. How many shapes did I hide from you?"

When general agreement has been reached, turn the piece over to check the results. Appoint record keepers to keep track of each example.

NUMBER OF SHAPES

WHOLE SHEET	THE SIDE WE SEE AFTER FOLDING	THE SIDE WE CAN'T SEE
9	4	5
9	3	6
9	1	8
9	7	2

This set of aids is quite versatile. Games can be built around the cards. Small groups of children can use them, individuals taking turns as folder, record keeper and question answerer. Still another dimension can be provided in the activity. "How many empty squares are there? How many with shapes? How many altogether?" Fold the card. "How many empty squares do you see? How many did I fold over?"

And the record grid gets bigger:

BEFORE FOLDING		AFTER FOLDING					
TOTAL SQUARES	TOTAL SHAPES	SHAPES WE SEE	SHAPES WE DON'T SEE	EMPTY SQUARES WE SEE	EMPTY SQUARES WE DON'T SEE	SQUARES WE SEE	SQUARES WE DON'T SEE
12	9	4	5	2	1	6	3
12	9	3	6	1	2	4	5
12	9	1	8	2	1	3	9
12	9	7	2	1	2	8	4

15. Counting and measurement

Pupils must learn the fundamental difference between *counting and measurement.* There are two variables which make the results of any measurement an *approximation:* (1) selection of the unit of measure; and (2) observation in comparing the unit with the object to be measured. All time spent on making these facts clear will be amply rewarded by a more thorough understanding of the problems involved in using the exact *language of counting* as the *language of measurement.*

In such questions as, "How many hands wide is your desk?" whose hand is meant? Is the thumb to be included? Is the hand to be squeezed together? What if the measurement doesn't come out even? Whose opinion is to be accepted?

Or in such a question as, "How many glasses in a quart?" what size glass is to be used? How full shall the glass be? What about spilling?

The purpose of having children participate in measurement activities is to establish for them the idea of: (1) selecting a unit, and (2) observing and recording an approximate comparison.

Explain to the class that in order for them to find out how many "shoe lengths" wide the room is, (stepped off toe-to-heel-to-toe) they first will have to decide whose shoes should be used to do the measuring. Children can match shoe lengths, and select whose shoes will be used in the measuring. Pupils with "approximately" the same length shoe can check the results.

Measurement is also used in "comparing" things that cannot be placed side-by-side. For example, "Which dimension of the room is biggest?" "Is the desk higher than it is wide?" "Are two first-grade rooms about the same size?" "Are two doors about the same height?" "Who lives closest to school?" "Are all city blocks the same length?"

16. Numbers have many names

On the board, draw the following chart:

"Numbers have many names!"				
other nomes for 3+2	other nomes for 10	other nomes for 11-8	other nomes for 7+5	other names for 9-9

If you draw an odd number of columns for *other names* on the board, select an even number of teams, preferably two. A member of the first team must write on the board *another name for 3 + 2* under the proper column. A member of the next team must write *another name for 10,* and so on.

The teams should alternate each time across. The team that had 3 + 2, 11 − 8, and 9 + 9 the first time around will have 10 and 7 + 5 the next time. Each team scores whenever one of its members give a

correct other name. Whenever one of its members gives an *incorrect other name,* the opposing team scores, provided one of its members spots the error.

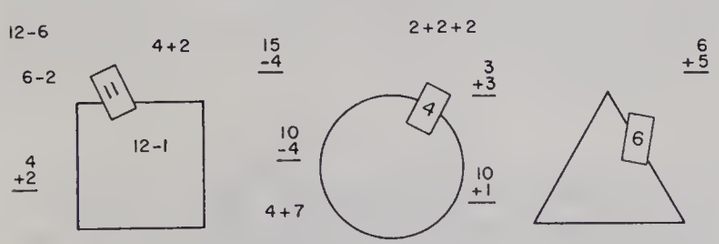
Games such as this let each child participate successfully at his own level. But most important, they drive home the fundamental concept that *numbers have many names.* This is the idea behind all *regrouping.* We can treat 32 as 20 + 12, so that 32 − 8 might be regrouped to read 20 + 12 − 8, and 20 + 4, and finally, 24.

Activities based on the idea that *numbers have many names* help the teacher to estimate the ability of each pupil. They help the teacher to discover who in the class is slow or quick, timid or confident. For example, Miss or Mr. *Way Out Ahead* may reveal themselves by using 3000 ÷ 1000 or $\sqrt{9}$ as other names for 11 − 8. (It is always possible to give the more apt pupils an opportunity to show off their prowess by asking them to write on the board the most difficult looking names for a given number that they can think of.)

A variation can be used to make the board game more difficult. A member of one team starts a name for the number in question, and the other team must complete it. Another name for 3 − 2 might be started as "0 + _." or "19 − _." The next player has to complete the name: "0 + 1," or "0 + 10 − 9," or "0 + 1 × 3 − 2," etc.

Sometimes, *you* may want to begin the other names in order to match the challenge of the situation to the recognized ability of the next player. This is important to the pupil: to be asked a question that is not too easy nor too hard, a question he can answer if he makes an effort.

A second activity. Draw this on the board:



As each expression is discussed, cross it out and rewrite it in the appropriate shape. Be as certain as possible that everyone understands what he is expected to do. Only if the child has no questions about the activity will you be able to judge successfully whether or not he is familiar with the arithmetic involved.

A third activity. Next, have class members tackle the following problem: "How many names for each of the numbers from 0 through 10, can you write in the form of ____ + ____ = ____?"

0	1	2
0 + 0 = 0	0 + 1 = 1	2 + 0
1 name	1 + 0 = 1	1 + 1
	2 names	0 + 2
		3 names

3	4	5

Then, have them record the results in a chart.

Number	Names in form of ____ + ____ = ____	
0	1	
1	2	
2	3	
3	4	
4	5	
5	6	
6	7	
7	8	
8	9	
9	10	
10	11	

This leads to a nice pattern. It suggests that there will be 21 names for 20 in the form of ____ + ____ = ____. Test the pattern. Does it keep on giving you reliable results?

A fourth activity. Next, you should explore the same question for names that are in the form ____+ ____+____=____.

0	1	2
0+0+0 = 0	0+0+1 = 1	0+0+2 = 2
(1 name)	0+1+0 = 1	0+2+0 = 2
	1+0+0 = 1	2+0+0 = 2
	(3 names)	1+1+0 = 2
		1+0+1 = 2
		0+1+1 = 2
		(6 names)

3	4	5
0+0+3 = 3	0+0+4 = 4	0+0+5 = 5
0+3+0 = 3	0+4+0	0+5+0
3+0+0 = 3	4+0+0	5+0+0
0+1+2 = 3	0+1+3	0+1+4
1+0+2 = 3	0+3+1	1+0+4
0+2+1 = 3	1+0+3	0+4+1
1+2+0 = 3	1+3+0	1+4+0
2+1+0 = 3	3+0+1	4+1+0
2+0+1 = 3	3+1+0	4+0+1
1+1+1 = 3	2+2+0	0+2+3
(10 names)	2+0+2	2+0+3
	0+2+2	0+3+2
	1+1+2	2+3+0
	1+2+1	3+0+2
	2+1+1	3+2+0
	(15 names)	3+1+1
		1+3+1
		1+1+3
		2+2+1
		2+1+2
		1+2+2
		(21 names)

The results, once more, are to be recorded in a chart.

Numbers	Names in form of ___+___+___=___	
0	1	
1	3	
2	6	
3	10	
4	15	
5	21	
6	?	

A fifth activity. Each of the following problems presents a pattern of its own:

1) "How many names for each of the numbers 1 through 10 can you find, using any one pair in one order only and omitting 0?"

0 → none

1 → none

2 → 1+1 (1 name)

3 → 2+1 (1+2 does not count, because 2+1 has already been included)
1+1+1
(2 names)

4 → 3+1
2+2
2+1+1 (4 names)
1+1+1+1

5 → 4+1
3+2
3+1+1
2+2+1 (6 names)
2+1+1+1
1+1+1+1+1

6 → 5+1
4+2
4+1+1
3+3
3+2+1 (10 names)
3+1+1+1
2+2+2
2+2+1+1
2+1+1+1+1
1+1+1+1+1+1

7 → 6+1
5+2
5+1+1
4+3
4+2+1
4+1+1+1
3+3+1
3+2+2 (14 names)
3+2+1+1
3+1+1+1+1
2+2+2+1
2+2+1+1+1
2+1+1+1+1+1
1+1+1+1+1+1+1

8 → (21 names)

9 → (29 names)

10 → (39 names)

Pupils will probably discover for themselves that it is helpful to devise a system for writing down the different names of each number. Otherwise it becomes difficult to keep track of the combinations. (Note: do not be disturbed if you cannot predict the number of combinations possible for 11. It took two famous mathematicians a long time to develop a formula that allows such a prediction to be made, and the formula is a complicated one.)

2) "Find all the names for the *even* numbers 1 through 10, using the sum of *even* numbers only and omitting 0:

4 → 2+2 (1 name)

6 → 4+2
2+2+2 (2 names)

8 → 6+2
4+4
4+2+2 (4 names)
2+2+2+2

10 → 8+2
6+4
6+2+2
4+4+2 (6 names)
4+2+2+2
2+2+2+2+2

3) Find the names for all the numbers 2 through 10, using the sum of *odd* numbers only, and omitting 0:

2 → 1+1 (1 name)

8 → 7+1
5+3 *(more, next page)*

3 → 1+1+1 (1 name)
4 → 3+1
1+1+1+1 (2 names)

5 → 3+1+1
1+1+1+1+1 (2 names)

6 → 5+1
3+3 (4 names)
3+1+1+1
1+1+1+1+1+1

7 → 5+1+1
3+3+1 (4 names)
3+1+1+1+1
1+1+1+1+1+1+1

5+1+1+1+1 (6 names)
3+3+1+1
3+1+1+1+1+1
1+1+1+1+1+1+1+1

9 → 7+1+1
5+3+1
5+1+1+1+1
3+3+3 (7 names)
3+3+1+1+1
3+1+1+1+1+1+1
1+1+1+1+1+1+1+1+1

	Names
2	1
3	1
4	2
5	2
6	4
7	4
8	6
9	7
10	10

17. Names for numbers in different forms. How many names could be written for zero in the next green box above 1, there are two possibilities. In completing this row of green boxes, a familiar pattern becomes apparent: one name for 0; two names for 1; three names for 2; four names for 3; five names for 4.

As observed earlier, a very different condition develops when the form becomes: "__ − __." Now there are unlimited possibilities.

Below the number line, the form "__ × __" appears. The pattern here is quite different: There are an unlimited number of names for 0 in the form "__ × __"; one name for 1 in the form "__ × __"; two names for 2 in the form "__ × __"; two names for 3 in the form "__ × __"; three names for 4 in the form "__ × __."

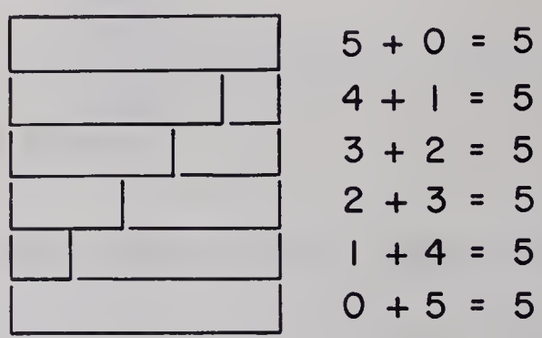
In this table, when there are only two names for a number in the form of "__ × __," the number is a prime number (a number that is divisable by exactly 2 different numbers, itself and 1): 2, 3, 5, 7, 11, 13, etc.

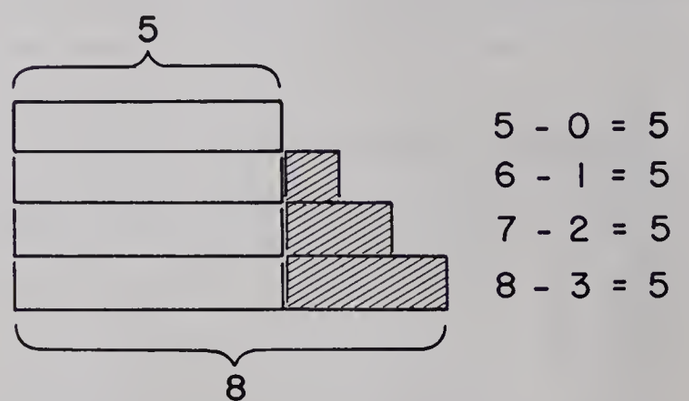
Finally, we discover that the form of "__ ÷ __" yields an unlimited group of names for each number. While there are three ways given here of writing these names, our main interest is in the form which leads so naturally into fractions: $\frac{1}{1}$, $\frac{6}{3}$, $\frac{3}{1}$, etc.

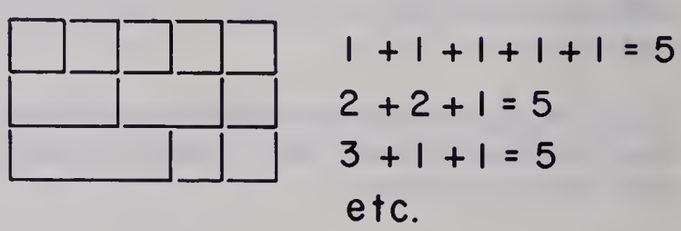
Care should be taken with other names for 0 in the form of division. It has to be asserted that division by 0 doesn't make any sense in arithmetic; it must be excluded. The only permissible names are 0 divided by any number other than 0: $\frac{0}{2}$, $\frac{0}{1}$, $\frac{0}{3}$, $\frac{0}{4}$, etc.

Manipulative materials. Give each student a series of paper or cardboard strips which he can manipulate. (A 1-inch square for one, a 1 × 2 strip for two, a 1 × 3 strip for three, etc.) Blocks or rods in a series of lengths representing 1 to 10, which you may already have in your classroom, will serve the same purpose. These can be used in a variety of ways, not only to reconstruct the sizes and relationships on these pages, but also to work out such problems as, "How many names has a number?"

For example, pupils can make "little pictures" of 5 in such ways as:







Fractions. "Is two dollars divided by two the same as one dollar?" ($\frac{2}{2} \rightarrow 1$) "If we divide 4 sandwiches by 2, do we have two sandwiches? Are 6 sandwiches divided by 3, two sandwiches?" ($\frac{4}{2} \rightarrow \frac{6}{3}$) "Would one sandwich divided by two be the same as 3 sandwiches divided by 6, or 2 sandwiches divided by 4?" ($\frac{1}{2} \rightarrow \frac{3}{6} \rightarrow \frac{2}{4}$)

If the dime is used as a basic unit, a nickel is to be thought of as $\frac{1}{2}$ a dime, a quarter as $2\frac{1}{2}$ dimes, a half dollar as 5 dimes, fifteen cents as $1\frac{1}{2}$ dimes. Continuing with the dime as the basic unit, three nickels would be called $3\frac{1}{2}$ dimes, and a quarter and a nickel would be $2\frac{1}{2}$ dimes and another $\frac{1}{2}$ dime."

In terms of dimes, the following could be accurately interpreted: $\frac{2}{2} = 1$; $\frac{1}{2} + 2\frac{1}{2} = 3$; $\frac{3}{2} = 1\frac{1}{2}$; $2\frac{1}{2} = \frac{5}{2}$.

Each time pupils answer questions like these, the mathematical symbol should be written on the board. The arrow, to be read "suggests" or "is the same idea as," is preferred in situations like this, to "=" ("equals"). The idea has not yet been firmly established that fractions are actually numbers or can be treated as numbers. Pupils do not know that they can add, subtract, multiply, and divide fractions. What pupils have been learning is that $\frac{4}{2} = \frac{6}{3}$; and that these are two names for the same number.

18. Row and column activities

The game, "Row-Column," can be adapted to any level of maturity. Use a "magic square," for example.

	A	B	C
X	8	1	6
Y	3	5	7
Z	4	9	2

A player for "Row" chooses X, Y or Z, and writes it down secretly. Then a "Column" player selects A, B or C. The players announce their choices. The number written where the announced row and column cross becomes "Row's" score.

Next a player for "Column" starts by choosing A, B, or C and secretly writing it down. Then a "Row" player selects X, Y, or Z. "Column's" score is discovered by finding the number where the choices of the two players cross.

Each person or team keeps a cumulative score. The first to reach a predetermined total, such as 25 or 50, wins.

Other grids can be made by adding, subtracting or multiplying the number in each cell of the magic square by one constant number:

Add 2

10	3	8
5	7	9
6	11	4

Subtract 1

7	0	5
2	4	6
3	8	1

Multiply by 2

16	2	12
6	10	14
8	18	4

In the examples above, the numbers in the rows and columns have been selected so that the sum for any three numbers down or across is the same. Neither "Row" nor "Column" is favored; chance for a high score is even. You may use smaller numbers on the square:

	A	B	C
X	1	2	1
Y	0	1	3
Z	3	1	0

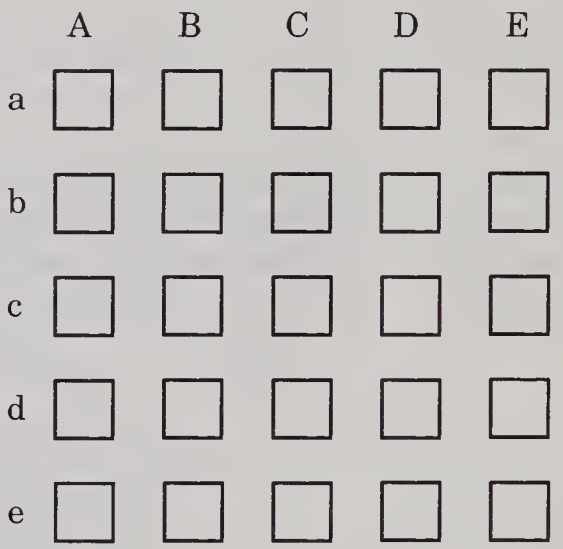
	D	E	F
U	0	1	2
V	3	0	0
W	0	2	1

	G	H	I
R	1	2	3
S	5	0	1
T	0	4	2

19. Exploring the magic square

More capable children may want to explore the principle of the magic square. In the first magic square shown above, the rows, columns and principal diagonals each total 15. If 6 were added to each cell, the new magic number would be 33; if each cell were multiplied by 3, the magic number would be 45.

Using notations. Put a diagram such as the following, on the board:



The notations B (AB,a) would refer to the number of boxes in colums *A* and *B* that are also in row *a* or, 2. A diagram such as the one above might represent seats in a classroom.

As soon as a pupil understands the idea clearly, he can look at a notation such as, N (ABD,ce) and know that the number N stands for is 6, because an array of 3 columns and 2 rows includes 6 items.

N(BC,ab) + N(D, abc) = 7 because 4 + 3 = 7. Later on, this can be interpreted as $(2 \times 2) + (1 \times 3) = 7$. Please do not suggest to pupils that this is an example of algebra. It is nothing of the kind. It is simply a kind of shorthand used in talking about a picture, such as the one on the board. Algebra is also a shorthand, but it is a different kind of shorthand.

Whenever you can, use notations in talking about a picture, an example, or a problem. Just remember that any notation is of use only when an agreement has been made about the meaning of the notation. Such notations (or shorthand signs) as "6," "10," "+," "×," "=," etc. are very useful because there is general agreement about what they mean in an arithmetical context.

More grids. Using any letters you want, create a grid similar to the following:

	a	b	c	d	e	f	g
h	B	B	B	E	E	E	E
i	B	B	B	E	E	E	E
j	F	F	F	E	E	E	E
k	F	F	F	P	P	P	P
l	F	F	F	P	P	P	P
m	F	F	F	P	P	P	P
n	F	F	F	P	P	P	P

According to the above grid, "N (ab,ijk)" means that the number of letters in columns *a* and *b*, and in rows *i*, *j* and *k* is 6. (There are 2 B's and 4F's.) A chart, based on the above grid, could be drawn:

	Total number of letters	Number			
		B's	E's	F's	P's
N (ab,ijk)→	6	2	0	4	0
N (cde,h)→	3	1	2	0	0
N (fg,mn)→	4	0	0	0	4
N (bcd,lmn)→	9	0	0	6	3

To vary the exercise, fill in the five columns with appropriate figures and ask pupils to give a notation that could be written in the left-hand column. For example:

?	8	2	4	2	0
?	4	1	1	1	1

Additional activities can be given to the children that finish the page first. This frees the teacher for individual work. Another grid that might be used is of shapes:

	a	b	c	d	e	f
g	□	□	□	△	△	△
h	□	□	□	△	△	△
i	•	•	•	△	△	△
j	○	○	○	○	○	○
k	○	○	○	○	○	○

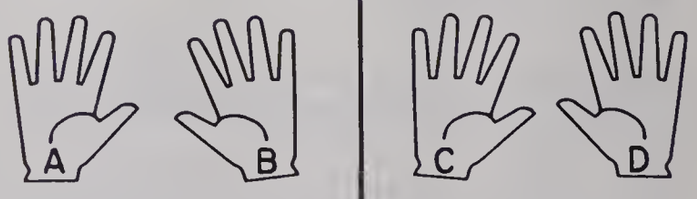
20. Multiplication activities

The series of activities that follows could be repeated in many different ways. Select the examples you like best, and repeat them often. Through these experiences, pupils will gain a real understanding of "multiplication."

a. Pairing digits. Put the chart below on the board. Fill in the "Digit on Left" and "Digit on Right" columns, as you go. Pupils should give you the answers to put in the other columns.

Digit on Left	Digit on Right		Total
2	0,3	20, 23	2
1, 2	1, 2	11, 12, 22, 21	4
1, 2, 3	5	15, 25, 35	3
7, 8, 9	1, 2	71, 72, 81, 82, 91, 92	6
1, 2, 3,	4, 5, 6	14, 15, 16, 24, 25, 26,	9
		34, 35, 36	

b. Pairing gloves. State the problem in this way: "Mary's mother bought her two pairs of gloves that were alike. In how many ways can she put them on?"



AC, AD, BC, BD

"If Mary lost one, in how many ways can she make a pair of gloves? Suppose she lost B?" (AC and CD.)

"Suppose she lost C, as well as B?" (AD only.)

"Suppose she found B and lost A as well as C?" (None—for she would have two gloves for the same hand.)

c. Couples. Here is another problem statement: "There were three men and three women sitting together. Jim said his parents were in the group. He wanted John to guess which couple his parents were. One way for John to find out would be to try all the possible combinations. But to do this he would have to give the men and women letter names in order to make a chart of the possibilities. The chart would look like this:

Men	*Women*
Mr. *A*, Mr. *B*, Mr. *C*	Mrs. *X*, Mrs. *Y*, Mrs. *Z*

"The possible combinations would be: *AX, AY, AZ, BX, BY, BZ, CX, CY, CZ*

"If John knew that Mr. *A* was not Jim's father, how many possibilities are there? (6-*BX*, *BY*, *BZ*, *CX*, *CY*, *CZ*.)

"If John knew that Mrs. *X* was not Jim's mother, how many possibilities are there? (4-*BY, BZ, CY, CZ*.)

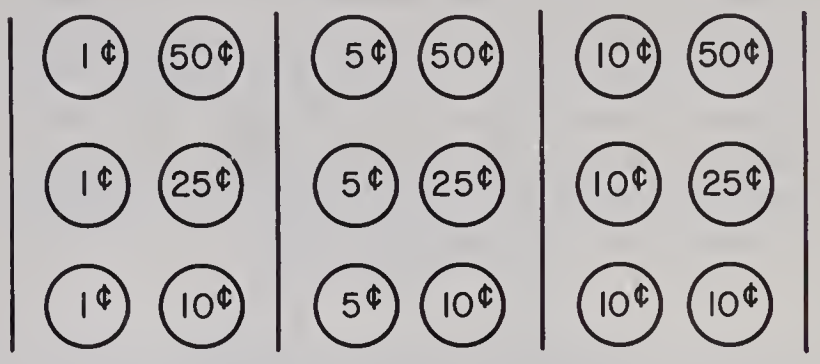
"If Jim said, 'Mr. *C* is not my father,' who would his father be?" (Mr. *B*.)

"If Mrs. *Y* is not Jim's mother, who is his mother?" (Mrs. *Z.*) "Who are Jim's parents?" (Mr. *B* and Mrs. *Z.*)

d. Pairing coins. You need two boxes and some coins. In one box put one penny, one nickel, and one dime. In another, put one dime, one quarter, and one half dollar. Have individuals from the class close their eyes and pick coins from the boxes. Make a record of the coins and total amount of money each child has picked:

Name	Box A	Box B	Total
Mary	1c	10c	11c
John	5c	25c	30c
Bill	10c	50c	60c
Barbara	5c	10c	15c
Alex	10c	10c	20c
Jean	5c	50c	55c
Bob	1c	50c	51c
William	10c	25c	35c
Helen	1c	25c	26c

The above chart includes the nine combinations that are possible. After you have recorded the first nine draws, ask if there are any repetitions. If there are, eliminate them and continue as before. When the class has come up with six or seven unique combinations, ask, "Can you think of any other possible combinations that we haven't had yet?" Pupils may find it easier to find combinations if they make a systematic record of the possibilities:



Repeat this same activity with fewer than three coins. Keep reducing or changing the number of coins—2 and 3, 3 and 2, 2 and 2, 1 and 2, 2 and 1, 1 and 1, 0 and 3, 2 and 0. When a record (table) is finished, erase the amounts in column A and B. Ask, "Can we tell which coins each child drew if we know only the total amount?" (Of course!)

e. How many combinations? This time, you must—Take three boxes. Place a penny and nickel in A; a nickel and dime in B; a dime, quarter, and half dollar in C. Have pupils draw a coin from each box. There are 12 possible combinations (because $2 \times 2 \times 3 = 12$) and the total amount represented by each draw will be different or unique. The following chart will result (do not include the right-hand column until the idea of multiplication has emerged):

A	B	C	Combinations	Because
2	2	3	12	$2 \times 2 \times 3 = 12$
2	2	2	8	$2 \times 2 \times 2 = 8$
2	1	3	6	$2 \times 1 \times 3 = 6$
2	1	1	2	$2 \times 1 \times 1 = 2$
1	2	1	2	$1 \times 2 \times 1 = 2$
1	1	3	3	$1 \times 1 \times 3 = 3$

If you use coins in the pattern suggested, every amount will be unique, so the selections can be reconstructed from a column showing only the amounts.

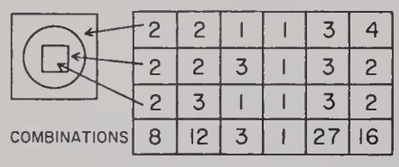
f. How many ways to color? The question here is: How many ways are there to color the design below, if the following conditions are met:

A—The area between the square and circle is to be red, blue or green.

B—The area inside the circle is to be red, blue or green.

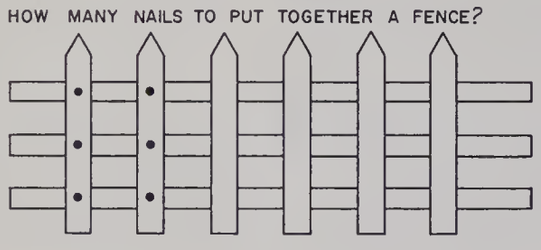
C—Both areas must be colored.

There are nine ways (because $3 \times 3 = 9$). Repeat the activity using three colors and two colors (result —6 combinations). To make the activity more difficult, increase the number of colors: three colors and four colors, for example, which produce 12 possible combinations. This activity can be extended to a design having three areas:



	2	2	1	1	3	4
	2	2	3	1	3	2
	2	3	1	1	3	2
COMBINATIONS	8	12	3	1	27	16

g. How many nails? Here is a seventh activity: Ask the class to tell you how many nails it is going to take to put together a fence like this:



Cross Pieces	Uprights	Nails
3	2	6
3	3	9
3	4	12
3	5	15
3	6	18

h. How many combinations of money? And an eighth—This example is planned to help develop the idea of multiplication and, at the same time, to provide pupils with an exercise in writing amounts of money. Ask pupils how many different amounts of money could be made by combining one each of the following paper money and coins:

Bills for $1.00, $5.00, and $10.00 and coins for 1c, 5c, 10c, 25c, 50c (Answer: 15, because 3×5=15)

$1.01	$1.05	$1.10	$1.25	$1.50
$5.01	$5.05	$5.10	$5.25	$5.50
$10.01	$10.05	$10.10	$10.25	$10.50

Bills for $1.00, $5.00, and $10.00 and coins for 5c, 10c, 50c (Answer: 9, because $3 \times 3 = 9$)

Try examples in which there is either no bill or no coin. In such cases there can be no combinations. This should serve to suggest the idea that $0 \times 3 = 0$, $2 \times 0 = 0$, etc.

i. How many batteries? From the world of sports—A baseball team has four pitchers and four catchers. How many different pitcher-catcher combinations can they plan on using? (A pitcher-catcher combination is called a "battery." Another team has four pitchers, and three catchers. How many combinations *less* than the first team can they make? (Four less—$4 \times 4 = 16$, $4 \times 3 = 12$ and $16 - 12 = 4$.) The chart below records the number of possible combinations that can be made if the number of pitchers is from 0 – 4 and the number of catchers if from 0 – 4.

	Pitchers				
Catchers	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	6	8
3	0	3	6	9	12
4	0	4	8	12	16

j. How many combinations of fingers? The question: "In how many ways can you combine a finger of your right hand with a finger of your left hand?"

Make the grid shown below. Ask a pupil to make a combination. Record it. Next, ask a child to make a different one. Record that. Etc.

	Right Hand				
Left Hand	Thumb	Fore	Middle	Ring	Little
Thumb					
Fore					
Middle					
Ring					
Little					

There will, of course be 25 combinations.

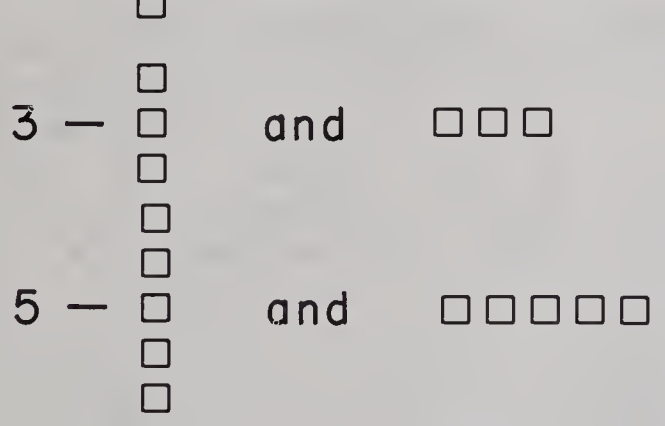
"What happens if we don't count thumbs?" (Only 16 combinations.) "Suppose we don't count thumbs or little fingers?" (Only 9 combinations.) Whenever a finger is eliminated from *each hand,* the number of possible combinations is still the square number of the fingers on any one hand: 16, 9, 4, 1, 0 (4×4, 3×3, 2×2, 1×1, 0×0).

k. Column and row arrangements. The problem is: Can you arrange 12 chairs in rows and columns so that each row has the same number of chairs and each column has the same number of chairs?

Total Chairs	In each Row	In each Column
12	3	4
12	4	3
12	2	6
12	6	2
12	1	12
12	12	1
16	4	4
16	8	2
16	16	1

If 24 chairs were used, it would be possible to illustrate: 1×24, 24×1, 12×2, 2×12, 8×3, 3×8, 6×4, 4×6. Other illustrations of multiplication can be shown by varying the number of chairs.

"In compliance with the rule that the number of chairs in each row and column be equal, what number of chairs would have to be set up so that there was only 1 row or 1 column?" (2, 3, 5, 7, 11 13, 17, 19, 23, etc.)



"These numbers are different from others. They are called prime numbers." Ask, "Can you find more *prime numbers*?"

l. How many first flights? You are a travel agent: In planning a trip, Mr. Traveler wanted to leave from New York, Boston, or Washington, D.C. He wanted to visit London, Paris, or Rome first. In how many ways could he plan his trip?

	Leaving from:		
Arrive	NewYork	Boston	Washington, D.C.
London	NY - L	B - L	W - L
Paris	NY - P	B - P	W - P
Rome	NY - R	B - R	W - R

Vary the number of cities Mr. Traveler might want to leave from, and the number of cities he might want to visit first.

m. How many combinations of food and drink? A menu:

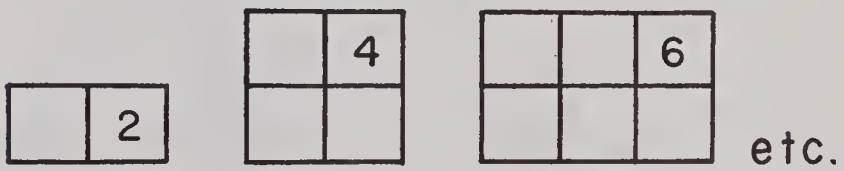
Sandwiches—	Hamburger	25c
	Tuna Fish	19c
	Hot Dog	17c
	Cheese	11c
Beverages—	Regular Milk	10c
	Chocolate Milk	13c
	Soda	6c

Jim had a sandwich and a beverage. His bill was 25c. What did he have? In how many ways could he have ordered a sandwich and a beverage from the same menu (12)?

	Reg. Milk 10c	Choc. Milk 13c	Soda 6c
Hamburger 25c	35c	38c	31c
Tuna Fish 19c	29c	32c	25c
Hot Dog 17c	27c	30c	32c
Cheese 11c	21c	24c	17c

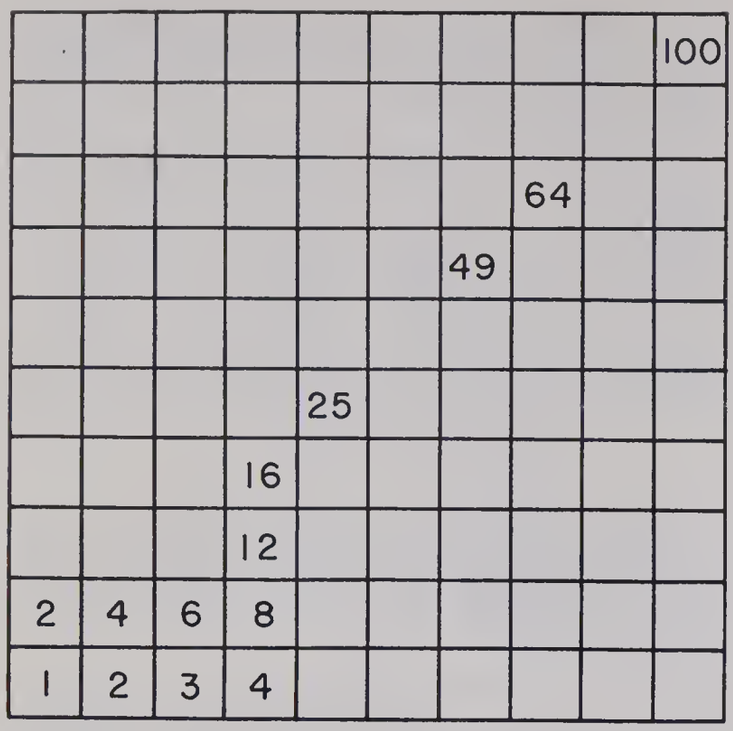
Vary the menu in numbers of items and prices. Be careful not to use prices that result in the same amount for two different combinations. One easy method is to add or multiply all the items by one figure. In this way you preserve the pattern of uniqueness worked into the example given.

n. How many squares? With scissors, you make a machine: From paper that is squared on one side, cut as many rectangles as possible, each containing a number of squares. No side should be more than the length of ten squares.



In the upper right-hand corners, record the number of squares. Arrange them in order of size. Stack them so the bottom left corners are all together—putting the one with the largest number down first, as shown at the top of the next column.

Wherever there is a square without a number, a rectangle can be cut to fill the gap. (You may want to start with all rectangles whose sides are not more than the length of five squares—so that 25 will show in the upper right-hand corner.) You have created what we call a *multiplying machine.*

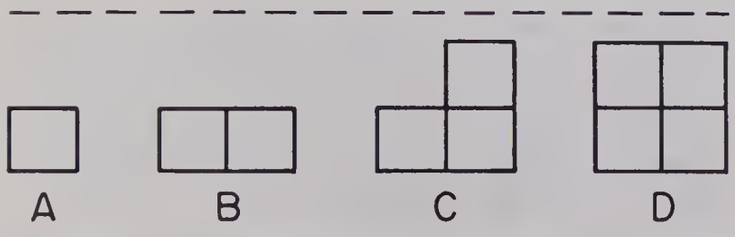


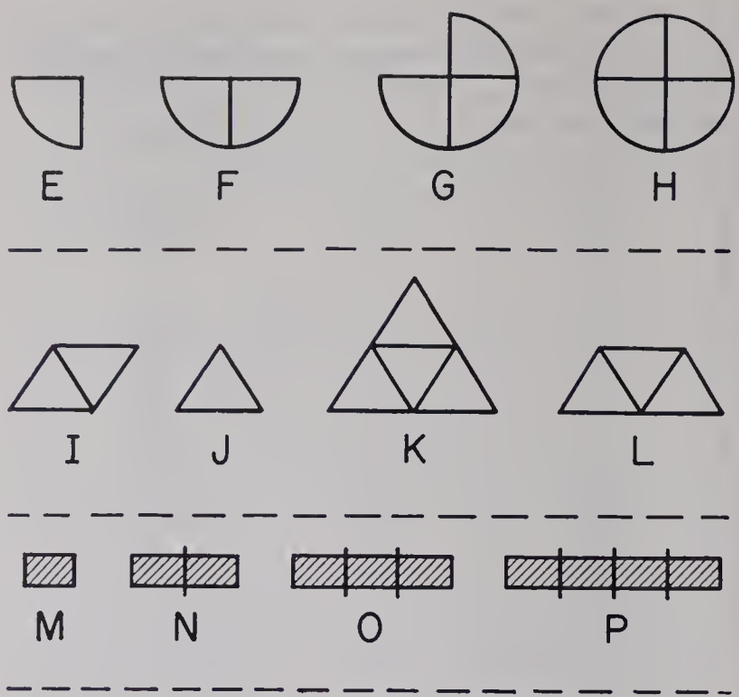
Put the rectangles face down on a table. Ask, "Who can find the rectangle divided into 4 parts? Who can find the 9 squares, 16 squares, 25 squares, etc. What shape are these parts?" (Squares; and again we come upon the square numbers—0, 1, 4, 9, 16, 25, 36, 49, etc.) Teams can play a game. One team says, "Find 24." If the other team finds "24," it belongs to them. If not, the first team gets a chance to find it. Each team collects the rectangles it has correctly picked. Score is kept by adding the *squares* of the rectangles each team has collected.

Each team collects the rectangles it has correctly picked. Score is kept by adding the *squares* of the rectangles each team has collected.

o. Shapes as units in multiplication. We know that "*If* . . . , *then* . . ." is the most used expression in mathematics. "*If* John has ten marbles and loses seven, *then* he must have three left." "*If* I draw a line from a point inside a triangle to a point outside a triangle, then I must cross the triangle." "*If* 19 is more than a first number and that number is more than a second number, *then* 19 is more than the second number." "*If* we agree that the size of a shape such as a certain square suggests five, *then* a shape twice as large suggests 10."

Many similar activities can be developed around diagrams drawn on the board:



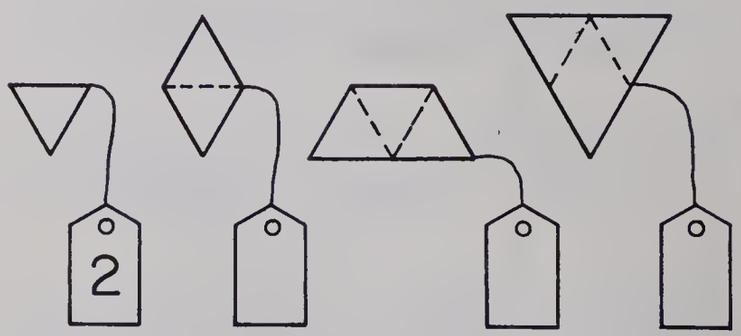


"*If* the size of A suggest five cents, *then* what is suggested by the sizes of B, C, and D?" B could suggest two nickels, one dime, or ten pennies.

"If the size of H suggests a dozen eggs, then what do the sizes of E, F, and G suggest?"

The designs used to reflect relative sizes may be any shape, any order, but not confusing.

Next, draw this diagram on the board:

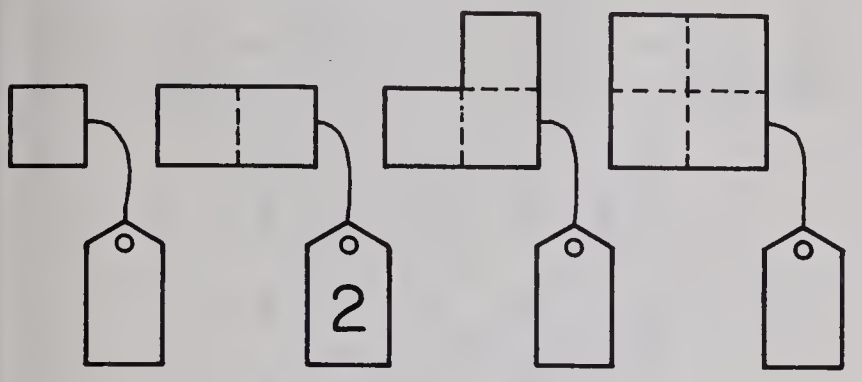


"*If* we say that the size of the first shape suggests *two* as the label indicates, *then* how would we label the other shapes?" The tags would read *2, 4, 6,* and *8*. Next, erase the *2,* the *6,* and the *8,* leaving the *4*. Ask pupils, "Suppose to start with we were given the number 4 in the second tag. How would we number the other tags?" The same system of labeling would result: *2, 4, 6,* and *8*. Repeat the activity, starting with only the tag labeled *6*; then with only the tag labeled *8*.

Next, erase all labels and put *6* in the second tag. "If the tag on the second shape were marked with a 6, *then* how would we label the tags for the other shapes? Is there a shape twice as large as the one we

have labeled with a 6?" (Yes) "What should its label be?" (12) "Now, how do we label the other tags?" (3 and 9) By this time, most of the class will have the idea and will be able to proceed with the page. Avoid any temptation to verbalize the relationship between shapes in some way such as: "The smallest tag should be half of the next one." Either a child has understood the activity, or he has not. If he has, then he can label the pictures properly. If not, he can't label them. There need be no verbalization to determine this question.

If some children are still confused, draw another diagram on the board, such as:



Repeat the development as already outlined. There is a subtlety in this activity that may properly disturb some children. They are used to counting *things*; a dot is a dot, regardless of its size. Therefore, it is necessary to make it clear that the ground rules have been changed. Emphasize the fact that we *are* concerned here with size. Just as it is possible to arrange collections of things according to the number of objects in them, it is also possible to arrange objects according to their size.

We have gone even further than a simple consideration of grouping objects according to size, by introducing the idea of *relative size.* If children have not absorbed this idea from the examples, keep drawing more pictures, more illustrations. The fact is that the child who is distrubed by the change in *rules* may be the most mathematically inclined. In any event, it is wise to accept the responsibility for introducing the new idea by taking great care in making it clear to the pupils.

Another possible approach is, "Suppose we have some pieces of paper of different sizes:

"Here are some small blocks. Now, how many blocks do you need to cover each piece of paper?" Keep a record of the results. Then put the blocks away.

"Suppose we had blocks of all the same size. If two of them would just cover the next-to-the smallest piece of paper, then how many would be required to cover the other pieces of paper?"

Don't by-pass or overlook the slowest child. The ability to make an agreement involving relative sizes is important in later problems of multiplication, area, numeration, fractions, etc.

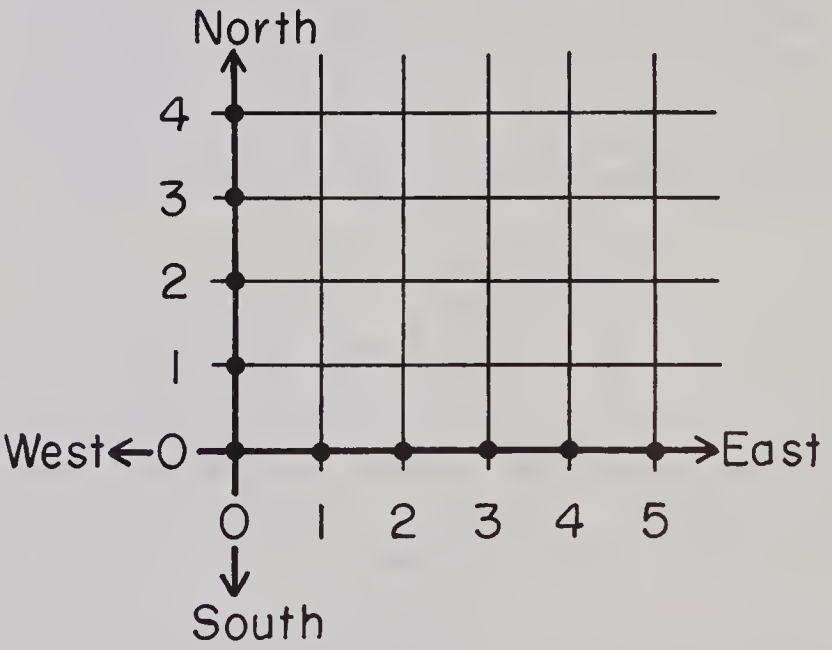
p. Using grids as a multiplication activity. Draw on the board two number lines that intersect, as shown below. Label each line beginning with "0" at the point of intersection.

Ask pupils, "How many jumps is it from 2 on one line to 3 on the other? How many jumps is it from 4 on one line to 4 on the other?" Record the results.

Starting point on one line	2	4	3	2	1	4	3
Landing point on the other	3	4	2	2	4	0	3
Jumps (of 1 each) required	5	8	5	4	5	4	6

Next, draw lines through each point on both number lines and introduce the game, *Taxicab Arithmetic.*

"Let's agree that each line we draw is a street on which taxicabs can travel. Of course, they must stay on the streets, but they can travel in any direction. The distances between intersections are *blocks.*

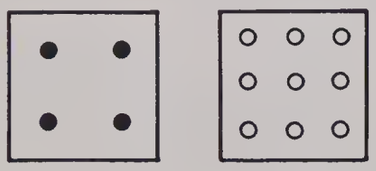


"We will call the West-East number line, W-E *Main Street,* and the South-North number line, S-N *Main Street.* Suppose a taxicab starts at 2 on S-N Main Street; it travels four blocks east and two blocks south. Where does it end up? How many blocks did it travel?" (It ended up at 4 on W-E Main Street, and it went six blocks.) "Could it have gotten there by a shorter route?" (No) "If it had stayed on

one of the Main Streets all the way, would it have traveled more blocks?" (No)

Try many trips from points on one Main Street to points on the other. Always, the shortest way is the same as if the trips were made only on the Main Streets. This game develops the two-dimensional concept that enables us to use the same diagram as a suggestion of multiplication. "Select any corner or intersection *not* on a Main Street. Draw a line from that point to take the shortest trip to each Main Street. How many square blocks are surrounded by those two trips and the two Main Streets?"

q. Square arrays. Sometimes it is possible to arrange things in a square array. For example:



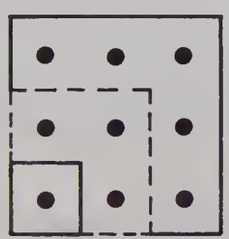
In the above pictures, four dots have been arranged in two columns and two rows; and nine dots have been arranged in three columns and three rows.

The following pictures:

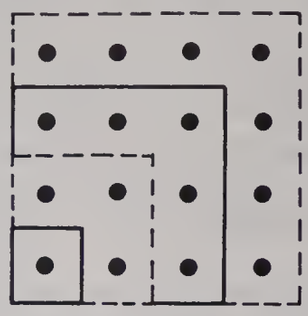
suggest it is possible to arrange 0 things in 0 rows, and one thing in one row. The question is, "What numbers of things can be arranged in this way?" (0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, etc.) One way to suggest how this particular series of numbers grows is to begin with one row and one column.

Then arrange two rows and two columns of dots, for example:

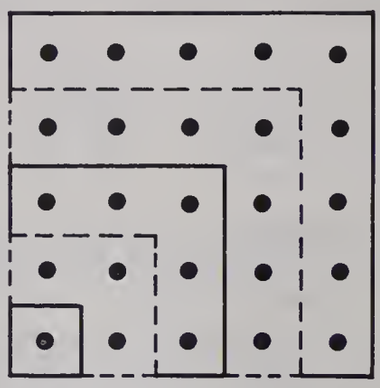
To one dot we have added three dots . . . and $1 + 3 = 4$. Next make three rows and three columns of dots:



To one and three dots we have added five dots . . . and $1 + 3 + 5 = 9$. Continue by making four rows and four columns of dots.



To one and three and five dots we have added seven dots . . . and $1 + 3 + 5 + 7 = 16$. Next:



To one and three and five and seven dots, we have added nine dots . . . and $1 + 3 + 5 + 7 + 9 = 25$. The above series of examples result in a pattern that can be easily shown in a table. Since order in addition does not change the results, reverse the order followed above to make the pattern more obvious:

$$1 = 1$$
$$3 + 1 = 4$$
$$5 + 3 + 1 = 9$$
$$7 + 5 + 3 + 1 = 16$$
$$9 + 7 + 5 + 3 + 1 = 25$$

"How can we extend this pattern?"

$$11 + 9 + 7 + 5 + 3 + 1 = \_\_\_\_ \ (36)$$
$$13 + 11 + 9 + 7 + 5 + 3 + 1 = \_\_\_\_ \ (49)$$
$$15 + 13 + 11 + 9 + 7 + 5 + 3 + 1 = \_\_\_\_ \ (64)$$

"How far can the pattern be extended?

r. Factoring and prime numbers. If a sum or a product is given, we can always find at least one pair of addends or factors, respectively, and, sometimes, many pairs of addends or factors. This can lead us to an investigation of how many pairs of whole numbers can be the factors for specific numbers.

Every number can be the product of itself and one, taken as a pair of factors. In every case, except 1×1 = 1, this particular pair of factors will be two different numbers. "Let's study several examples and find all the pairs of facts there are for each."

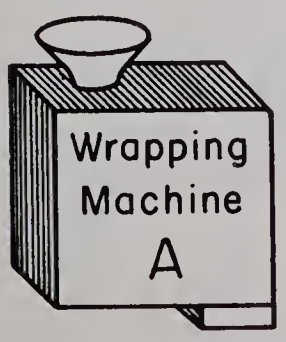
Number taken as a product	Itself and "1"	Other pairs
0	(0,1) (1,0)	(0,7) (0,99) etc.
1	(1,1)	none
2	(2,1) (1,2)	none
3	(3,1) (1,3)	none
4	(4,1)(1,4)	(2,2)
5	(5,1) (1,5)	none
6	(6,1) (1,6)	(2,3) (3,2)
7	(7,1) (1,7)	none
8	(8,1) (1,8)	(2,4) (4,2)
9	(9,1) (1,9)	(3,3)
10	(10,1) (1,10)	(2,5) (5,2)
11	(11,1) (1,11)	none
12	(12,1) (1,12)	(2,6) (6,2) (3,4) (4,3)

Continue this chart as far as time permits, and keep a separate list of all numbers that have two *different* factors, itself and *one*, but for which there are *no other pairs*. (Use of the word *different* is deliberate, because we want to exclude *one* from this list. It will be more useful to us later, if we agree that *one* is not to be considered as a prime number.) The list of numbers having *no other pairs* is:

2, 3, 5, 7, 11, 13, 17, 19, etc.

We have met this sequence of numbers before. We call them *prime numbers* because as products there is only one pair of two different factors that name them: themselves and one. Are all *prime numbers* also odd numbers? (No, there is one exception: 2.)

s. Wrapping machine arithmetic. For this activity, you first make a sketch of a crude machine on the board, labeling it "Wrapping Machine A." Any sketch will do, as long as it has a "place to put things in" and a "delivery station" or output.



Input	Output: Number of packages	Output: Contents of each package
6 marbles	2	3 marbles
18 cookies		

"We don't know what's inside this machine. We only know what we put in the machine and what comes out of it. If we put six marbles in through the funnel, the machine gives us two packages, neatly wrapped. When we open the packages, we find that each contains three marbles, what kind of machine is this? Let's try it again. We put in 18 cookies. What do you suppose the machine delivers?" Of course, it might wrap them in packages of three each. Or, it might wrap them in two packages.

"Actually, it gives us two neatly wrapped packages. How many cookies do you suppose we'll find in each package?" Nine, of course.

"Now, let's see if we know what the machine will do in the following situations:

Input	Output: Number of packages	Output: Contents of each package
6 marbles	2	3 marbles
18 cookies	2	cookies
30 pennies		
a dozen eggs		
ten trucks and four cars		
a half-dozen apples		
two dimes and six pennies		
three nickels and five pennies		
		7 pencils
		25 stories
		11 letters of the alphabet

The fourth item may cause a feeling of doubt. Some pupils may think the machine couldn't handle such a problem any better than it could handle one marble . . . or one nickel. But most pupils will agree that the machine would give us two packages with six eggs in a package.

With ten trucks and four cars as the problem, the machine might wear a puzzled look—if it had one to wear. The machine might put two cars and five

trucks in each package. That would be consistent. Or, it might put seven vehicles in each package: one containing four trucks and three cars; the other containing six trucks and one car.

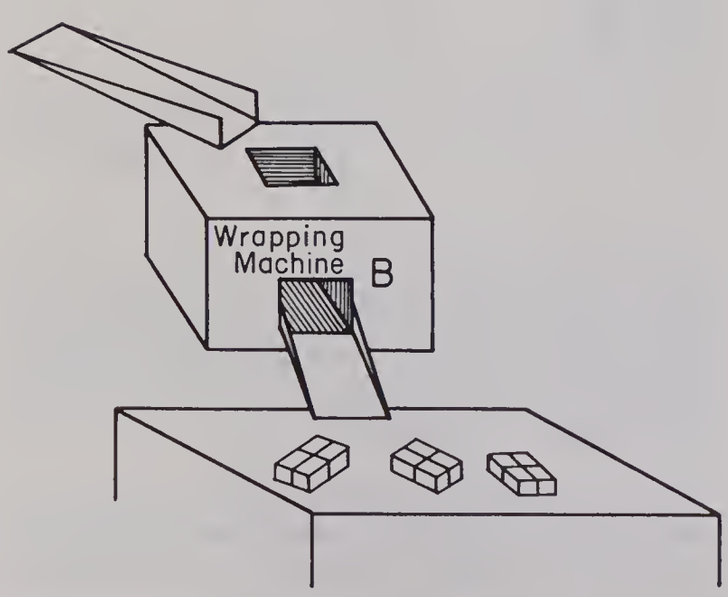
Next item—easy.

Next item—"two dimes and six pennies." Does the machine recognize that a dime is worth 10 pennies? Well, in one package there were four pennies. "What do you suppose was in the other?" Two dimes and two pennies. There were four coins in each package. Evidently, the machine simply counts the objects, and separates them into two groups with the same number in each group.

"Suppose we know the contents of one of the packages. Can we know anything about the 'input'?" If one package contains seven pencils, then the input must have been 14 objects, seven of which were pencils. It is possible that there were 14 pencils, but it is also possible that there were 11 pencils and three erasers, etc.

"What could we feed into the machine that would stump it completely?" Certainly one quarter (25c), one egg, three eggs, etc.

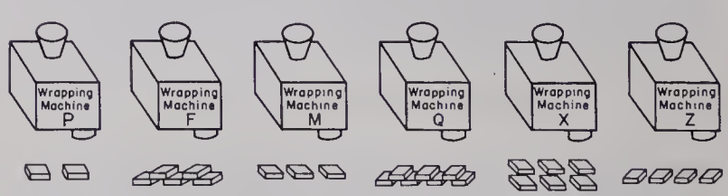
Draw another wrapping machine, "Machine B," that is identical in operation, only it always delivers three packages. If you were to feed it six marbles, you would get them wrapped neatly in three packages, with two marbles in each. Have pupils keep a record of output and input. "How many of the examples of input, that we have considered could both machines handle?" (Six marbles, 30 pennies, a dozen eggs . . . but not 10 trucks and four cars, nor two dimes and six pennies.)



	Output	
Input	Number of Packages	Number items in each package
6 marbles	3	2 marbles
30 pennies	3	10 pennies
1 doz. eggs	3	4 eggs
a set of tricycle wheels	3	1 wheel
1 penny, 4 nickels, 2 dimes, 5 quarters		4 coins
2 doz. eggs		
		7 marbles
		4 nickels, 2 pennies
		3 trucks, 3 cars
		11 stars

This same activity can be extended with machines that deliver four, five, six, etc., packages with the same number of items in each package.

After the idea has been sufficiently developed, draw a series of machines and a series of input-output records on the board. "There has been a mix-up, and the tables (or records) have been separated from the machines they refer to. Can you match each machine with its record?"



Machine M		
Input	No. Pkg.	Items ea. Pkg.
12 coins		
		8 eggs
	3	2 whistles
30 books		
		7 pencils

Machine Q		
Input	No. Pkg.	Items ea. Pkg.
14 marbles		
		10 beads
35 cookies		5 cookies
21 cars		

Machine F		
Input	No. Pkg.	Items ea. Pkg.
10 wheels		
5 apples		
15 cents		
40 stamps		
		4 pears
		$\frac{1}{2}$ doz. eggs
		10 marbles

Machine P		
Input	No. Pkg.	Items
		7 books
		3 rulers
8 pears		
3 doz. bananas		18 bananas

Machine X		
Input	No. Pkg.	Items
1 doz. eggs		a pair of eggs
12 trucks		
		5 books
		10 erasers
		1 apple, 3 oranges

Machine Z		
Input	No. Pkg.	Items
6 prs. shoes		3 shoes
2 doz. Oranges		
		$\frac{1}{2}$ doz. eggs
		7 marbles

Of course, the first record must belong to "Wrapping Machine M"—because the number of packages is shown.

The clue for the second machine is given by the information that it wraps 35 cookies in a certain number of packages—and there are five cookies in each package . . . so, there must be seven packages—and that chart must belong with Machine Q.

Similar clues are given, with varying degrees of directness, for all others except the record that belongs with Machine F. This can be solved in at least two ways: (a) since all other records have been assigned to a machine, the unassigned record and machine must belong together; or (b) assuming the machine can successfully handle all the inputs listed in the record, it must wrap each in five packages because the machines that wrap 2, 3, 4, 6, or 7 packages could not handle all the inputs successfully.

Next, we have enough information now to complete each line of each record. And, we could expand each record as far as we like.

To summarize these results, another chart can be drawn:

	Machine P		Machine M	Machine Z	Machine F	Machine X	Machine Q
Number of Items in Input	In Pkg.	Each	Pkg.	Pkg.	Pkg.	Pkg.	Pkg.
2	2	1					
3			3 1				
4	2	2		4 1			
5					5 1		
6	2	3	3 2			6 1	
7							7 1
8	2	4		4 2			
9			3 3				

Should the chart be extended through inputs of 20 or more, interesting patterns will be detectable. In this chart, if a machine cannot handle a particular "input," the space pertaining to it is to be crossed out. If the class enjoys the "wrapping machine," continue with the idea by asking about this "can't handle" situation. "Machine M wraps every input in three packages. If you fed seven marbles into it, the machine would be stumped. Could a machine be designed to solve this problem some way?" Consider every answer as a possibility. Would the output be confusing? One possible answer would be to have a machine designed so that it would mark the third package, or "only one" or, "There's only one—a full box would be three." This answer anticipates "division remainders," but our main attention is still focused on the idea of multiplication and real life situations where it has application.

Each episode of this "wrapping machine" story is as open-ended as you want to make it. Each fact of multiplication can be introduced at least once and repeated as often as required. Individual work can be easily assigned. "Make up a chart of your own that might accompany Machine X that has an output of six packages. List 10 inputs and their appropriate outputs. List 10 inputs the machine couldn't handle." The problems can be applied to the other machines.

21. Pairs of answers

A question such as, *How many are 3 and 5?* that can be answered off-hand with a number is usually a trivial question and appropriate only for drill. A question such as, *How many are 357 and 294?* may test the pupil's ability to *regroup* or to *carry*, but the answer again ends with a number: 651.

"Pairs of Answers" is an activity that requires

thinking, but it does not involve large numbers. A notation that may be new for many children is used in asking a question that involves "pairs of answers." For example:

This is an open sentence. It is neither true nor false. The box and triangle are to be considered as *reserved seats* for pairs of whole numbers. If "1" approaches the box or square, and if its partner is "3," they can both sit down because 1+3 = 4. Another pair that would be permitted to take these reserved seats is "0" and "4." In fact, any pair can be seated, provided their sum is 4. A record of pairs with a sum of 4 can be made as follows:

	A	B	C	D	E	F
□	1	0	3			
△	3	4	1			

The pairs mentioned above have been recorded under A and B in the chart. Under C another pair has been recorded. Are there more? Yes, "4" in the square and "0" in the triangle; also, the pair composed of "2" and "2." *That is five pairs. Are there more?* No!

This activity does not involve large numbers, but the questions it raises are not trivial. Pupils are given an opportunity to record the suitable pairs in a pattern that makes an eloquent answer to the question, *Are there any more pairs?*

0 and 4
1 and 3
2 and 2
3 and 1
4 and 0

22. Chain reactions

Given a rule and a pair of numbers, it is possible to set off a *chain reaction* that can be continued across the page, and on indefinitely.

Sample rule: "Each number is to be the sum of the two numbers immediately to the left of it. Given:

1 2 _ _ _ _ _ _ _ _ _ _

the next term must be 3, because 1 + 2 = 3; the next term is 5, because 2 + 3 = 5; the next is 8, because 3 + 5 = 8; etc. Thus, this chain reaction gives us:

1 2 3 5 8 13 21 34 55 89

and it could keep on going. Change the given numbers, and another chain reaction is under way.

A chain reaction can be worked in both directions. Let the rule remain the same, but change the given numbers and place them mid-point of the reaction:

_ _ _ _ 10 16 _ _ _ _ _ _

Under the rule, the number to the left of 10 must be 6, because 6+10 = 16; the number to the right of 16 must be 26, because 10+16 = 26. So, extended in both directions, the results are:

2 2 4 6 10 16 26 42 68 110 _ _

Another sample rule: *Each number must be twice as large as the one to its left.* Now it takes only one given number to start the action.

1	2	4	8	16	32	64
Given	(2×1)	(2×2)	(2×4)	(2×8)		
128	256	512				

Another sample rule: *Each number must be the difference of the two numbers above it: the one on its left subtracted from the one on its right. The bottom row is always the same number.*

1					
	2				
		3			

Told that the bottom row is all alike, it is possible to begin the reaction:

1	A				
	2	B			
		3	3	3	

If 2 in the second line must be gotten by subtracting 1 from A (or A–1 = 2), then A must be 3, because 3–1 = 2. By the same reasoning, B–2 = 3, so B must be 5 because 5–3 = 2. Completed, the chain reaction would look like this:

1	3	8	16	27	41	58	78	101	127
	2	5	8	11	14	17	20	23	26
		3	3	3	3	3	3	3	3

These chain reactions get underway with a mini-

mum of direction and *go someplace.* They require accuracy, since each in turn depends on its predecessor and any little error along the way will make every succeeding term wrong. Children get a real feeling of satisfaction in being able to demonstrate this accuracy.

23. Fractions

In the material that follows the various expressions used in speaking of fractions will be studied briefly. No rules or procedures should be discussed until pupils have first had an opportunity to acquaint themselves with the whole body of knowledge regarding fractions. Before rules or procedures can be meaningful, this knowledge must be worked into sensible and understandable patterns that are consistent with all the arithmetic of whole numbers pupils have already learned. The approach to fractions used in the *Math Workshop for Children* is to consider this topic not as a separate idea, but rather as an extension of the idea of number, from whole numbers to those numbers which men invented so that there would be an answer to every division problem. Without fractions, $2 \div 3 =$ ____ and $3 \div 2 =$ ____ have no meaning; there is no way to complete these mathematical sentences. We explore fractions to see if they behave like the numbers we know. We examine them to find the answers to these, among other, questions:

1) Can fractions be arranged from small to large?

2) Can pictures be drawn to suggest the idea in the context of a picture that also suggests whole numbers?

3) Are there fractions that are other names for whole numbers?

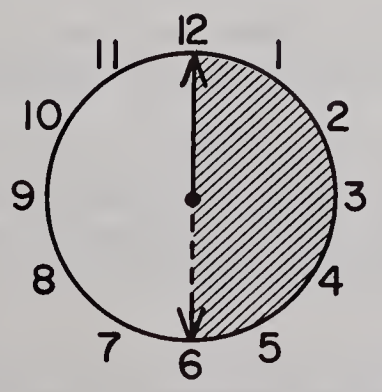
4) Is there at least one fraction to name each whole number?

5) Can fractions help name points on the number line?

6) Can fractions be added? can they be subtracted?

7) Can fractions be multiplied and divided?

Activity 1. Draw a clock on the board, with the minute hand pointing to 12. "Where will the minute hand be in half-hour?" Have a volunteer draw an indication of the hand's second position.



"Color in the area the minute hand covered as it traveled from the first to the second position. How much of the clock face is colored? Have we divided the clock face into two parts about the same size?"

Start over with the minute hand pointing to 12. "Where will the minute hand be in a quarter of an hour? Where will it be a quarter of an hour later? Indicate both positions. Where will it be half an hour from the starting time? Where will it be three-quarters of an hour from the starting time? Indicate this new position. We have shown four of the positions the minute hand passes. Color in the part of the face the minute hand passes over in a quarter of an hour." (If colored chalk is available:) ... "with another colored chalk, shade another part of the face that the hand would pass over in a half-hour."

Follow a similar discussion with one-third and two-thirds of an hour; also with one-sixth, two sixths, three-sixths, four-sixths, five-sixths of an hour.

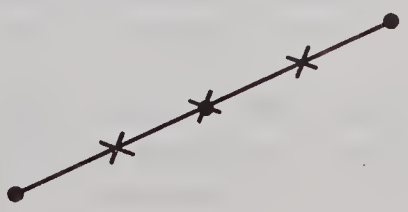
Activity 2. Put a square on the board. "Can someone draw a line that divides the square into 2 parts about the same size? Can someone else draw a different line that divides the square into 2 parts? Any more lines?" (Any straight line drawn through a point where the two diagonals would cross is a correct response. Some child may draw a zig-zag or curved line that would have to be accepted.

Activity 3. Draw a line on the board—preferably at a slight angle—and put a dot on each end.

"This is my picture of a piece of wire. The ends are sharp and I don't want to get scratched, so I've covered them. Where should I cut the wire to make two pieces the same length? Mark the point with a dot.

"Suppose I wanted to cut the wire into four pieces that are the same size. Can you mark with a cross the places where I ought to cut it?

"Now I'm going to draw another sketch of my piece of wire. This time I will bend it a little bit.

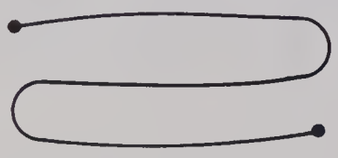


"Where should I cut the wire if I wanted to have two pieces the same size? Can you mark with a dot the place where I should cut it? And where would you put cross marks to indicate where I should cut the wire if I want to have four pieces the same size?

"I found a longer piece of wire."

"Where should I cut it, if I want to have two pieces of the same size? Where should I cut it, if I want to have four pieces the same size?

"Now I've bent the wire into an S-shape.



"Now I put two sharp bends in it.

"Finally, I bend it into something like a circle.

"In each instance, mark with a dot the place where I should cut the wire if I want to have two pieces the same size, and then mark with crosses the places where I should cut if I want to have four pieces the same size.

"I finally cut my wire into two pieces.

"No sooner had I done this than I was sorry. I had really wanted to cut it into four pieces that were the same size . . . but I only have two. What can I possibly do?" (You will have plenty of advice.)

"If I change my mind again after I have cut the wire in four pieces, is there anything I can do?" (You'll have more advice: solder two pieces, twist them together, glue them, nothing, get another piece of wire, etc.)

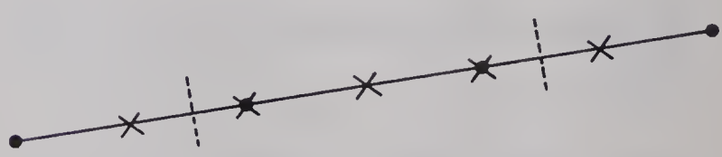
We have carefully avoided using the word "equal." This expression is, as previously explained, to be used in referring to the relationship between two names for the same number: $3 + 9 = 12$. Besides, in dealing with wires or sketches, it is not possible to cut them or divide them into "equal" pieces. If we remind pupils that what we really mean when we say "of the same size" is "*about* the same size," there won't be any misunderstanding. "Remember, when we are concerned with answering a question of 'How many,' we must always be exact. But when we are concerned with answering a question that involves any kind of measurement and that asks 'How much,' we can never be exact."

Because the kind of sketches used in this activity are meant only to *suggest* fractions, answers need only be approximate.

Activity 4. "I have a rope. I've knotted each end so it won't unravel. If I were to lay it out straight, it would look something like this:

"If I wanted to cut the rope into three pieces about the same size, where should I cut it? Mark the places with dots.

"Now suppose I wanted to have twice as many pieces of rope, all about the same size. Using a little cross can you mark each place on the rope where I should cut it?



"Why do some of the dots and crosses mark the same place?" (Don't expect well formulated answers. Don't try to settle the matter.)

"If I wanted to have only two pieces about the same size, where would I cut? Is the place marked?

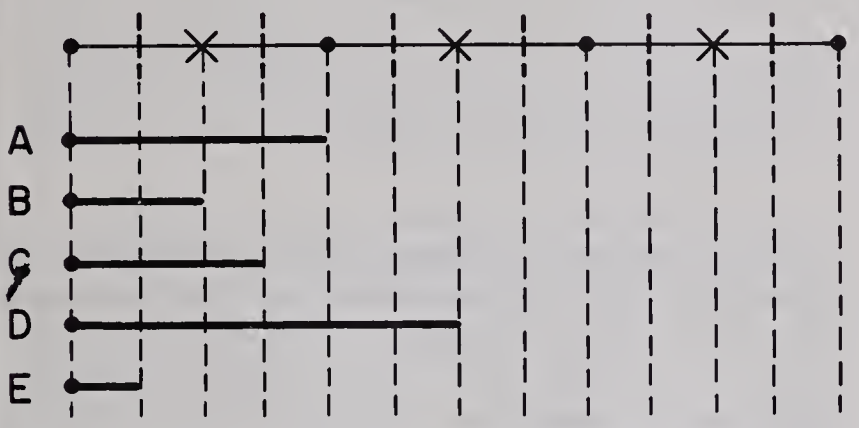
"Suppose I wanted four pieces about the same size? (Cut it in half—and then cut each of the two pieces in half.) Are all the places to be cut marked? Use a dotted line to mark the new places I would need to cut.

"Now, what do you suppose has happened? I've

changed my mind again. I want a dozen pieces – all about the same size. Where shall I cut?" Discuss this matter for a while. You might cut the rope in half and then cut each of those pieces in half to get four pieces. You would then cut each of the four pieces into three parts. Or you might cut the rope into six parts; and cut each of the six pieces in half. Or, you might cut the rope into three parts; and then cut each of those in half, and halve each of the six pieces; etc.

"Are any of the places to be cut already marked? How many more places do we have to mark?" (4) "Indicate these places with dotted lines."

On the board draw the following:



"At one time, I wanted to cut my rope in two pieces about the same size. Does any line on the chart look something like one of those pieces?" (The line marked D.)

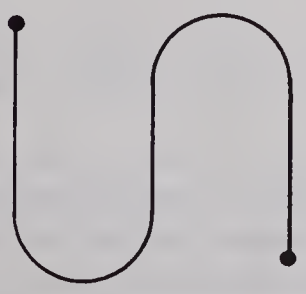
"What do the other sketches suggest to you?

"How many pieces of rope the size of line B would I be able to cut from a piece of rope the size of line D?

"Is the line marked A twice as long as the line marked C? Which line is twice as long as A?"

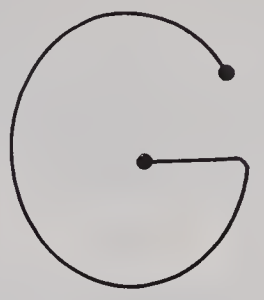
Shift the discussion to a pattern similar to that outlined for page 88 by "bending or twisting your rope."

"Now, I've twisted my rope until it looks like a letter S turned over on its side:



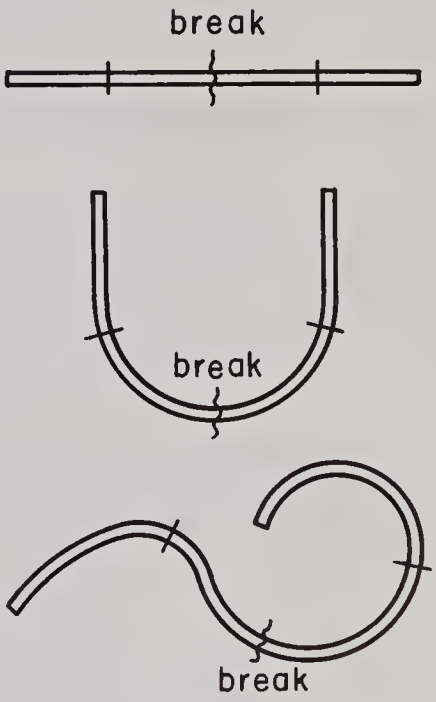
"Use dots to mark the places on the rope that should be cut, if I want to have three pieces about the same size. Next, use little crosses to show where I should cut the rope if I want six pieces about the same size." Do not go further with this example.

"Now, I have twisted the rope into the shape of the letter G."



Have the children mark the places to be cut if you want three and then six pieces.

Activity 5. "Let's talk about some taffy that Martha made. She pulled it into a long piece and layed it on the kitchen table. In what shape do you think she put it down? Well, she might have layed it out straight . . . or in the shape of the letter U . . . or in a variety of other shapes. Here are some possibilities:

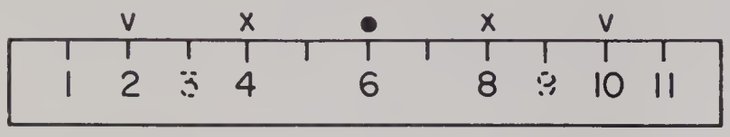


"Martha decided that she wanted to divide the candy into three parts: one for Billy, one for Jean and one for herself. So she marked the places she would break it when the candy got hard." Have pupils make the markings.

"Then Martha remembered that Billy had braces on his teeth and couldn't eat taffy. So, she decided to break it in half when it was hard. She marked the place. "Have a volunteer mark the place." When the candy was hard, Martha broke the large piece into smaller ones. Just then Billy came running into the kitchen. 'Look, Martha, no more braces. Oh, boy, you've made some taffy.' "Now what do

you think Martha did?" She could break each half into three pieces each so that she would have six pieces: two for Billy, two for Jean and two for herself. Or, she could break the taffy at the first marks she made and have two large pieces and two small pieces. Now are the two small pieces together about the same size as one of the large pieces?

Activity 6. Here is a picture of a one-foot ruler.



"Where is the middle of the ruler? What number is usually written at that point on a one-foot ruler? Use small crosses to divide the length of the ruler into three parts about the same size. What numbers would be written at these points?"

"What points would divide the ruler into six parts about the same size? Label the points as they usually are labeled on a ruler."

"Is it easy to divide the ruler into 4 parts about the same size? Label these points if they are not already labeled."

"Are there any more points to be labeled on this ruler?"

Activity 7. "In the shorthand of arithmetic, there are at least three different ways of writing the following sentence." On the board, write: *Six divided by two equals three.*

"How many ways can you write this statement in shorthand, using no written words at all?" The three ways that interest us are given below with 2 examples of each:

$6 \div 2 = 3$	$\frac{6}{2} = 3$	$2\overline{)6}^{\,3}$
$12 \div 4 = 3$	$\frac{12}{4} = 3$	$4\overline{)12}^{\,3}$

This suggests that the following statements are true:

$$6 \div 2 = 12 \div 4 \text{ and } \frac{6}{2} = \frac{12}{4} \text{ and } 2\overline{)6} = 4\overline{)12}$$

In each case, read these sentences: "Six divided by 2 equals 12 divided by 4."

"The shorthand $\frac{6}{2} = \frac{12}{4}$ suggests a game. The rules are simple. Make all of the following sentences true, using nothing but this shorthand in which to express division facts. We never use the numeral 1 unless we have to."

$$\frac{9}{3} = \frac{12}{?} \qquad \frac{16}{4} = \frac{?}{7} \qquad \frac{25}{?} = \frac{15}{3}$$
$$\frac{9}{3} + \frac{4}{2} = \frac{20}{?} \qquad \frac{15}{5} + \frac{12}{3} = \frac{?}{2}$$

If there is any question about these examples, write them each three ways. For instance:

$9 \div 3 = 12 \div ?$ and $\frac{9}{3} = \frac{12}{?}$ and $3\overline{)9} = ?\overline{)12}$
$(9 \div 3) + (4 \div 2) = 20 \div ?$
$\frac{9}{3} + \frac{4}{2} = \frac{20}{?}$
$3\overline{)9} + 2\overline{)4} = ?\overline{)20}$

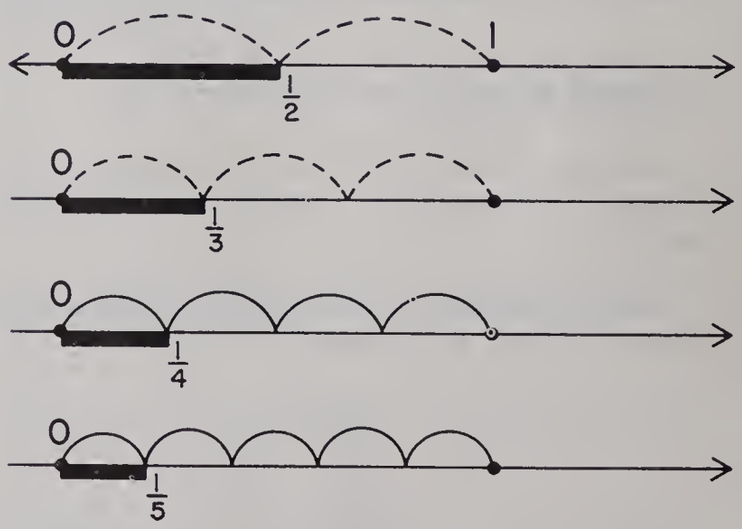
After pupils have grasped the idea of the game, move on to examples for which there are several solutions:

$\frac{18}{3} - \frac{18}{2} = \frac{?}{7}$ (because $6 - 4 = 2$);
$\frac{9}{?} + \frac{8}{?} = \frac{?}{2}$ (because $3 + 2 = 5$);
$\frac{?}{2} + \frac{?}{3} = \frac{?}{8}$; $\frac{8}{2} \times \frac{6}{3} = \frac{?}{6}$;
$\frac{20}{2} \div \frac{6}{3} = \frac{?}{5}$; etc.

As pupils play this game, they are on the firm and familiar ground of arithmetic. They will be asked to play the game often. And when they finally come to: $\frac{2}{3} + \frac{3}{4} = ?$ and $\frac{1}{3} \times \frac{2}{5} = ?$, pupils will not find these expressions mysterious. They will have a feeling that such expressions probably fit into the scheme of: $\frac{8}{2} + \frac{6}{3} = \frac{54}{9}$

Activity 8. "Indicate jumps on a number line that suggest;"

$1 \div 2$ or $1/2$	$1 \div 6$ or $1/6$
$1 \div 3$ or $1/3$	$1 \div 7$ or $1/7$
$1 \div 4$ or $1/4$	$1 \div 8$ or $1/8$
$1 \div 5$ or $1/5$	$1 \div 9$ or $1/9$



Have the children complete an example suggesting each of the expressions shown above. When confronted with an example such as $1 \div 4$ or $1/4$, discuss procedures that might be followed, such as: first find the point halfway between 0 and 1; then find the point halfway between 0 and this new point (or,

between 0 and 1/2). When preparing to draw a sketch to suggest the idea of $1 \div 6$ or 1/6, the first step might be the same as in the previous example, or it might be to indicate the points suggesting $1 \div 3$, and then to find the mid-points of each segment. In many cases, the work can begin exactly as for one of the other examples, and then carried further. In such cases, two examples will show their family resemblances. Since 1/6 is related both to 1/2 and 1/3, either of these can be used to begin the work. One-eighth is related in an even closer way to 1/2 and 1/4. One-ninth bears a family relationship to 1/3.

"What labels do not seem to be related to any others in the group under consideration?" (Only 1/5 and 1/7.)

"Do you suppose that 1/5 and 1/7 belong to any families at all?" (Yes, 1/5 is related to 1/10, 1/15, 1/20, etc.; and 1/7 is related to 1/14, 1/21, 1/28, etc.

Refer back to the diagrams suggesting 1/2, 1/3 ... 1/9. "If we want to label all the landing points between 1 and 0, what labels would you suggest in each instance?" (2/3, 1/3; 3/4, 2/4, 1/4; 4/5, 3/5, 2/5, 1/5; etc.)

"Starting from 0, which is larger, a jump to 3/5 or a jump to 1/2?"

"In the following pairs, circle the member that is located further away from 0 on the number line."

1/3, 2/3	3/4, 1/4
2/4, 1/5	2/3, 5/6
2/8, 1/5	1/2, 3/7

"Consider the following. Which is further from 0?"

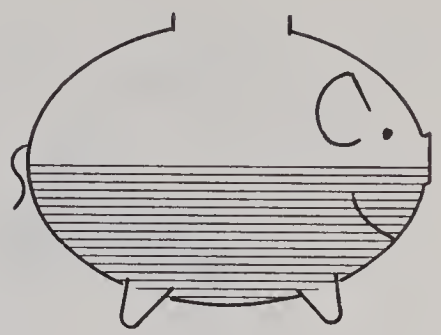
1/2, 3/6	2/4, 1/2
2/6, 1/3	2/3, 6/9

"We have a shorthand that expresses these ideas. I will write some true statements in this shorthand. Can you complete the others?"

$1/2 > 1/3$	$1/2 < 3/4$	$1/2 = 3/6$
$3/9 = 1/3$	$3/6 < 4/6$	$7/8 > 1/4$
2/4 3/6	2/3 2/4	1/7 1/4
1/3 2/7	3/5 1/2	4/8 3/6

Continue examples until everyone can use this notation. Then explain that ">" is shorthand for "is greater than . . ." and "<" is shorthand for "is less than . . ." Point out that if the shorthand symbols reminds one of an arrow, the thin or sharp-end points to the "smaller of the two," and the wide part of the symbol is close to the "larger of the two."

Activity 9. "Here is a sketch of a piggy bank. It is half-full of pennies."

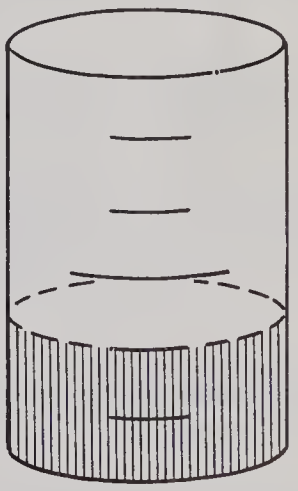


"Can you tell me how many pennies there are? What do you need to know before you can tell me the approximate number? If I told you how many pennies were needed to fill the piggy bank, could you tell me about how many pennies the bank holds when it is half-full?"

Value when Full	No. of Pennies when half-full
\$1.00	
\$ .50	
\$3.00	

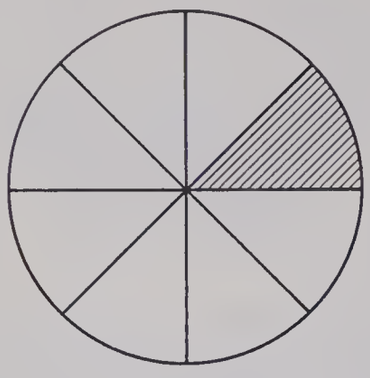
Much of mathematics can be stated in the form of "*If..., then....*" propositions. *If* a full piggy bank holds 150 pennies, *then* about how many pennies does it hold when it is half-full?

Activity 10. "Here is a water tank. The marks on it help to indicate how full it is."



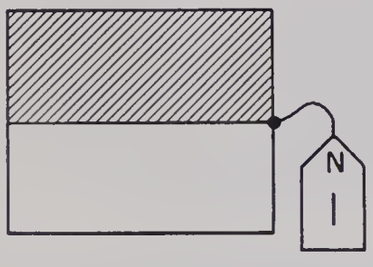
No. of Gallons when full	No. Gallons in shaded area
6	
60	
3	
1	

Activity 11. "Here is a pie that is cut into 8 pieces. There are 16 ounces in a pound. How many ounces would there be in a shaded piece?"



Weight of Whole pie	Weight of 1 piece	
In pounds	In pounds	In ounces
1	1/8	2
2		
4		
6		

Activity 12. "Here is a sketch of 1 of something. This is indicated by the 1 on the tag. What does the shaded area suggest? The unshaded area?"



Shaded	Unshaded	Tag
1/2	1/2	1
		2
		1/2

"Suppose we change the label or tag. Let's write 2 in the tag. Now, what do the shaded and unshaded areas suggest?" (1 and 1)

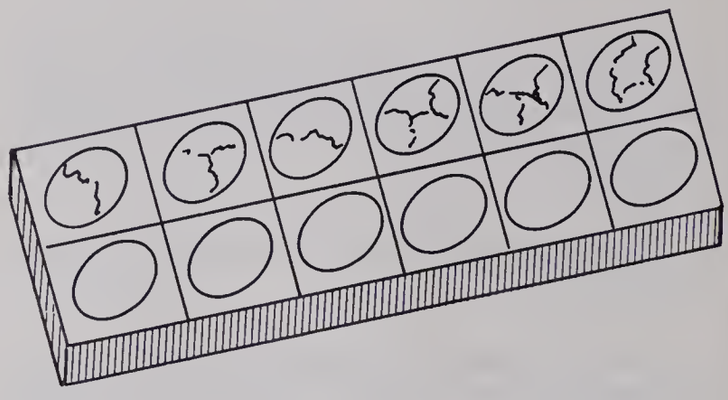
"Suppose we change the tag to read 1/2. Now, what does the shaded area suggest? (1/4 and 1/4) Without discussing, make the comment—"It seems that a half of one-half is one-fourth. In money this would mean that half of a half-dollar is the same as a quarter of a dollar."

"Can you make sketches suggested by each row in the following chart?"

Shaded	Unshaded	Tag
3/4	1/4	1
1/3		1
	2/5	1
3/2		2
	5/3	2

Don't hurry. Discuss the adequacy of each sketch, possible improvements, similar sketches that suggest the same idea, similar ideas suggested by the same sketch, etc.

Activity 13. "Bill said that 6 eggs in a carton of 12 were cracked. His sister said, 'But it looks to me as if 3 in each half-dozen are broken.' His little brother said, 'In each group of two eggs, one is broken.' When his Mother looked at the carton she said, 'Why half of them are broken.' "



"Here is a sketch of the situation. Who was right?" Write on the board:

Bill: "6 eggs are cracked."
Sister: "3 in each half-dozen are broken."
Little brother: "One out of every 2 is broken."
Mother: "Half are broken."

"Are they all saying much the same thing? In mathematics, we say that numbers have many names. We say, for example, that:

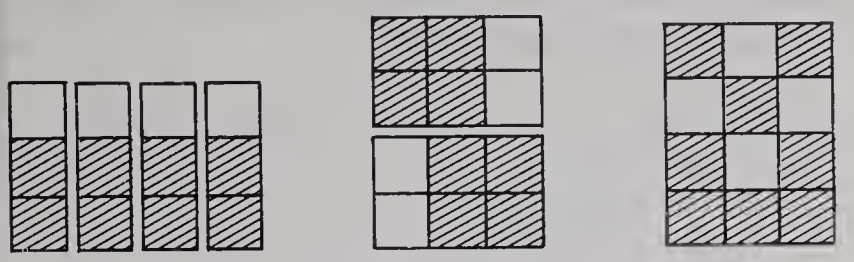
$$\underline{1/2} = 2/4 = \underline{3/6} = 4/8 = 5/10 = \underline{6/12}$$

"The underlined names for 1/2 suggest something quite similar to the remarks about the broken eggs."

Activity 14. Cut squares about 2 × 2" out of cardboard that is white on one side, gray on the other. Or have the quares colored one color on one side, another color on the other. Put 12 squares in a box. Mix them up and dump them out on a table.

"How many squares are white? How many are gray? Without turning any over, arrange them in

a 3×4 array. Arrange them in different patterns." (Unless the result is 5 of one color and 7 of another, they can usually be arranged and rearranged to indicate many ways of saying the same thing. Suppose there are 4 white and 8 gray:



grey – 2 out of 3 white – 1 out of 3

4 out of 6 2 out of 6

8 out of 12 4 out of 12

After several examples, add 4 more squares so that there are 16 in all. Now all even numbers of squares, either white or gray (if one is even, so must the other be even), can be arranged in more than 1 way. The most useful results for making our point here are 8 – 8 and 12 – 4.

"The shorthand we have been using: 1/3, 2/5, 4/8, etc., can be used to keep track of our results. In the example when 8 out of 12 were gray, we can express the idea in the sketch this way:

Gray	White	All together
8/12 or 4/6 or 2/3	4/12 or 2/6 or 1/3	12/12, 6/6, 3/3, 11/11, 10/10, 9/9, etc.

"We can read this new shorthand: '8 out of 12 are gray' or '4 out of 6 are gray,' etc." Write the following on the board, and have pupils complete them.

If 8/12 are gray, then are white?
If 3/12 are white, then are gray?
If are white, then 7/12 are gray?

"Are the following statements true?"

If 8/12 are gray, then 1/3 is white?
If 1/4 is white, then 9/12 are gray?
If 2/12 are white, then 5/6 are gray?

Add 8 more squares. Now there are 24 in all. Proceed in same general way.

Talk about each example. Draw sketches. Encourage the children to explain their arguments. Keep asking, "Does that sound reasonable? Does everyone agree?"

Do not expect to reach any general rules about this idea that, later on, will be referred to as "ratio." We are introducing the idea at this point mainly for discussion purposes. We will come back to it.

Use the 2 × 2" squares of white-gray cardboard (or equivalent) for an early experience with the idea of chance or probability.

Put two squares in a box. Mix them, and drop them on a table. "How many are gray? In what different ways can squares show up?"

0 out of 2 may be gray
1 out of 2 may be gray
2 out of 2 may be gray

"Are there any other possible outcomes? Which result do you think will happen most often?"

Let several groups experiment, and have them each keep a record:

	How many are gray?		
	0/2	1/2	2/2
Totals	6	14	4

The more trials each group makes, the more the results will approximate a 1-2-1 ratio. That's a secret.

Then make a compilation of totals.

Group	0/2	1/2	2/2
A	6	14	4
B	4	10	8
C	7	15	4
D	8	13	8
Totals	25	52	24

Expand the activity to investigation of results when three squares are used. "What are the possible combinations? How many can be gray each time?"

0 out of 3
1 out of 3
2 out of 3
3 out of 3

Again, keep records.

	How many are gray?			
	0/3	1/3	2/3	3/3
Totals	5	14	16	6

Again, have several groups keep track of as many trials as possible – and then record the totals as

shown above. (This time, the outcomes will approximate 1-3-3-1—*that's a secret.*)

The only goals we have at this point are to plant the idea that certain combinations seem to occur more often than others; and to use familiar notations to express ratios. (For example, 2/3, meaning "2 out of 3.") However, we do not yet refer to the term *ratio.*

Activity 15. Experiment with notations and sketches such as follows:

"Here are two arrays of 15 squares. On one, we will mark one out of every three squares with an A. On the other, we will mark two out of every five squares with a B. How many squares out of the 15 will be marked in each case?"

A	A	A	A	A

1 out of 3,
or 5 out of 15

B	B			
B	B			
B	B			

2 out of every 5,
or 6 out of 15

Next show the same number of squares marked A and B together on a single sketch of 15 squares:

A	A	A	A	A
B	B			
B	B		B	B

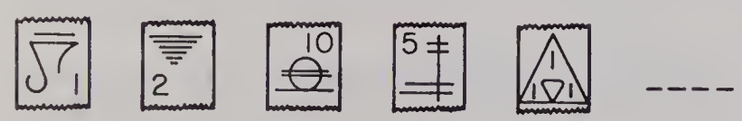
"If we were to simply put one diagram on top of the other, two squares would contain 2 marks. Since we want only one mark in any one square, we must shift two of the B's.

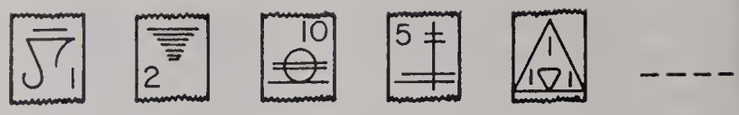
5 out of 15 and *6 out of 15* together are *11 out of 15*; or *1 out of 3* and *2 out of 5* together are *11 out of 15.*

"In shorthand, we might write:

5/15 and 6/15 → 11/15, or
1/3 and 2/5 → 11/15

Activity 16.
"Alice saved stamps. Her friend, Maxine, also saved stamps. Alice decided to give Maxine one out of each pair of duplicates. So, she arranged them in pairs across ther desk, something like this:





"How many did she give Maxine? Do you have enough information? Which of the following sentences need to be completed?

A. Alice has __ stamps arranged in pairs on her desk.
B. The stamps were worth $__.
C. She had __ pairs arranged on her desk.
D. Alice gave Maxine __ out of every pair of stamps.
E. Alice had __ stamps of which she had no duplicates.
F. Of those so arranged, she kept __ stamps.

Completions for any one of the sentences A, C, D, or F will provide enough information to answer the question conclusively.

Without completing A, C or F, have pupils draw a chart of some of the possibilities, if they know how many stamps Alice gave Maxine:

Number to Maxine	1	5	50		
No. of stamps arranged in pairs	2	10	100	etc.	

"In grocery stores, you will find cans, cartons, bottles and packages marked this way:

3/15	2/11	4/25
5/20	5/22	5/13

"What do you think those marks mean?" (3 for 15¢; 2 for 11¢; 4 for 25¢; etc.)

Activity 17. "If a can is marked 3/15, how much would you pay for 1 can?" Someone will say that if 3 cans cost 15¢, then 1 can would cost 5¢ because 15 ÷ 3 = 5. Someone else might contend that such is not the case, because then the cans would be marked 1/5. If the price is meant to indicate a bargain for purchasing 3 cans, then the price for 1 can must be 6¢. By purchasing 3 cans for 15¢ the customer saves 1¢ per can.

On the board draw a chart for each case:

No. of Cans	1	2	3	4	5	6	7	8	9	10
Price in cents	*5*	*10*	15			30				

No. of Cans	1	2	3	4	5	6	7	8	9	10
Price in cents	*6*	*12*	15	*21*	*27*	30				

"We agree that the price is 3 cans for 15c in either case. How much would 6 cans cost? How much would 9 cans cost? The same in both records? Now, how about 1 can? We have agreed that I can might cost 5c or 6c. In each case, how much would 2 cans cost?" (10c in the first case; 12c in the second.) But someone might argue that if 1 can costs 6c and 3 cans cost 15c, then you probably get a small bargain for buying 2 cans at a time. So, while 1 can cost 6c, 2 cans may cost only 11c. That would look like this:

		Savings
1 can	6c	0
2 cans	11c	1c
3 cans	15c	3c

So, we need another chart:

No. of Cans	1	2	3	4	6	7
Price in cents	*6*	*11*	15	5	30	

A final word

The *Math Workshop for Children* has not invented a new mathematics! Nor have its authors spun novel theories of education. Rather, in it they have tried to bring into a whole the most universally accepted ideas of both educators and mathematicians.

Basically, then, the *Math Workshop for Children* takes hold of traditional subject matter and prescribes for its study large doses of solid mathematics, rather than the traditional presentation of isolated facts.

We have accepted the challenge of mathematicians and educators to think about the problem long enough and with the creative interest it takes to distill mathematical ideas until their essence emerges in the simplicity that new young minds insist upon—and respond to.

Robert W. Wirtz
Morton Botel
W. W. Sawyer

Notes to accompany *Work Text* page 1

Work Text page 1 sets the tone for the never-ending search for patterns that ties the *Level F Work Text* together. In the top left corner, we take a look at multiplication of a rather unique group of numbers—.01, .1, 1, 10, and 100. All the products might be seen as a form of:

00010000

and the only problem is, "Where do I place the decimal point?"

Beneath this (perhaps diagonostic) exercise is an example discussed earlier. Notice each column: the number written in the top row is the sum of the numbers written in the blue- and green-tinted areas; the numbers written in the bottom row are products of those numbers. In the last column on the right, a question might arise: "Shall we write 20 in the green space or should we write 50 there?" There is no pattern to guide your selection, so it makes no difference, since $20 + 25 = 25 + 20$ and $20 \times 25 = 25 \times 20$.

The series of examples beginning "$2 \times 0 = 1 \times 1 - 1$" has an interesting outcome that we don't expect after a first casual glance. As it unfolds, we gain confidence that $31 \times 29 = 30 \times 30 - 1$ or 899—without even computing 31×29 in the usual way. In this example, as it often happens, study of a pattern leads to a useful shortcut for one whole class of examples.

In each chain reaction such as 1, 1, 2, 3, 5, 8, 13, etc., the secret is that each term is the sum of the two written on the left.

After the three chain reactions are well underway, someone may discover an easy way to find any term in the third row by noticing it always turns out to be the sum of the two terms written above it—if no errors crept in.

Four examples of Cross Number puzzles need to be completed. Even children who have never seen this unique device before can see what's going on—particularly if the arrows are taken as a hint. We might look at the first example in this way: (read, "3, 8 → 11" as "three and eight goes to 11")

Top row	3, 8 → 11
Bottom row	15, 9 → 24
Left column	3, 12 → 15
Right column	11, ____ → 24
Middle column	8, ____ → 9
Middle row	12, ____ → ____

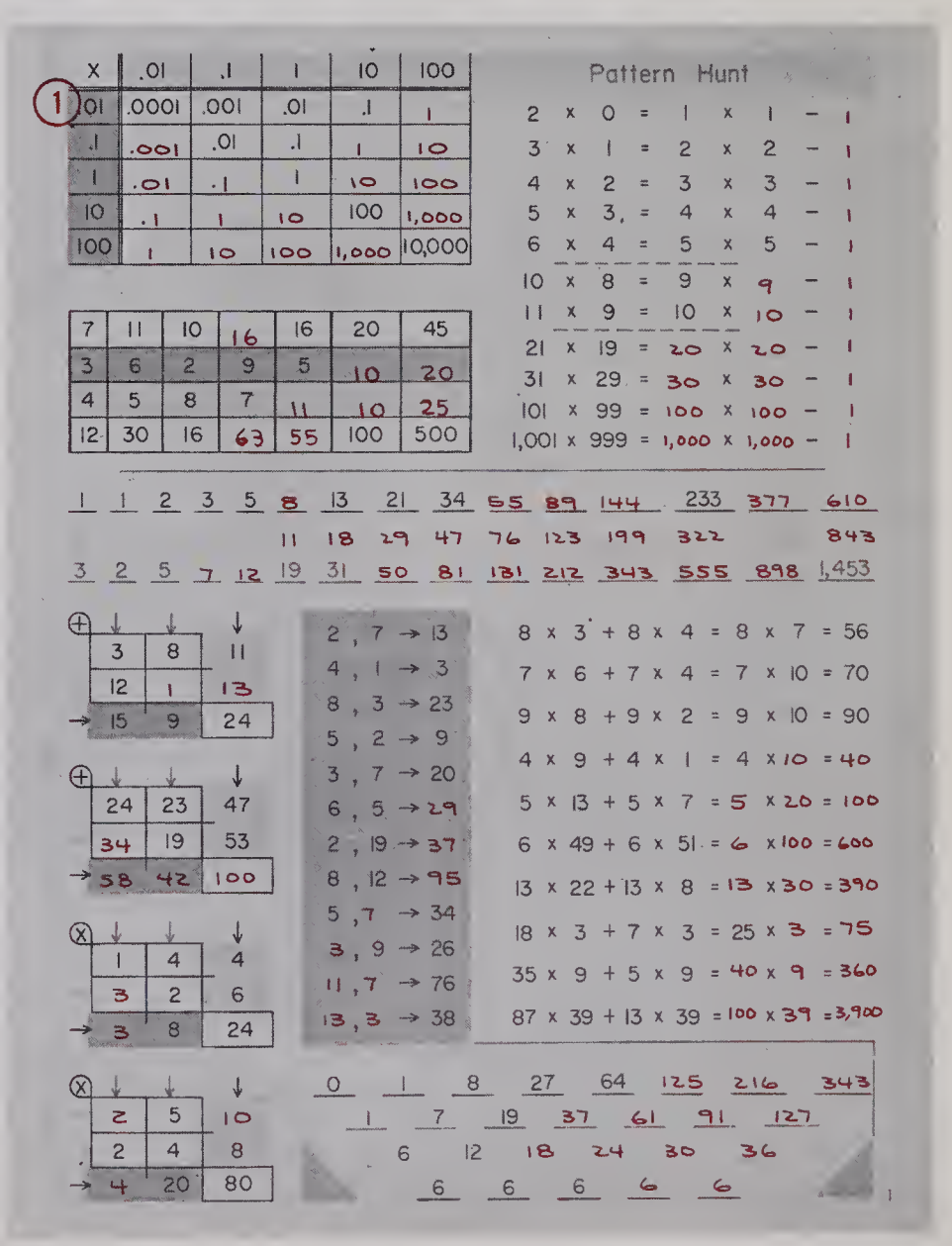
The "rule" is revealed in the first three items that are completed. We find that each column and each row is an example of addition. The top row of the second example is quite convincing that this puzzle follows the same pattern. And the top row of the third signals a change—because we know that $1 + 4 \neq 4$ (read, "one plus four does not equal 4"). But a list of the rows and columns, noting completed groups first, shows:

1, 4	→ 4
4, 2	→ 8
4, 6	→ 24
____, 2	→ 6
____, 3	→ 24
1, ____	→ ____

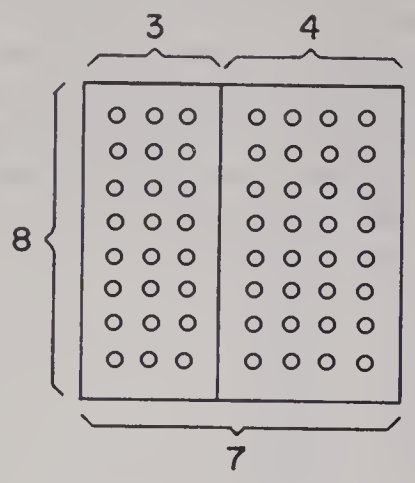
The first three suggest that "multiplication" has come in as a rule. And the completed rows and column of the last example suggests the same rule:

5, 4 → 20
2, 4 → 8

So, from the patterns above we are led to a conclusion that is borne out by the symbols in the upper left-hand corner of each example. (The secret in the center of the lower section is: multiply the numbers noted on the left and add 1 to the product; write the result on the right of the arrow.)



The idea on the right might best be suggested by a sketch or two:



$$8 \times 3 + 8 \times 4 = 8 \times 7$$

(The secret of the last pattern: each number written below the top line is the difference between the two numbers written above—one on the right and the other on the left:

$1 - 0 = 1$ $\quad$ $7 - 1 = 6$
$8 - 1 = 7$ $\quad$ $19 - 7 = 12$
$27 - 8 = 19$ $\quad$ $12 - 6 = 6$)

Notes to accompany *Work Text* pages 2-7

A "million" is a word or a symbol spoken or written on paper. It's a "big number." "It's an awful lot of money." "A million miles is about four times the distance between the earth and our moon."

On *Work Text* page 2, there is a sketch that suggests what a million blocks might look like. Some children may want to count the bottom row on each side—and the number of layers. They will find 100 by actual count of those rows and layers.

How many in each layer? $100 \times 100 = 10{,}000$. There are 100 layers—$100 \times 10{,}000 = 1{,}000{,}000$.

At the top of *Work Text* page 3, a section of 10 layers is illustrated. If children are skeptical, let them count. On that top layer they can count 10,000 squares—100×100. In 10 layers—as shown—there would be $10 \times 10{,}000$ squares, or 100,000 squares.

The next sketch shows a section of 10 layers in which each layer contains 10×100 or 1,000 blocks $= 10 \times 1{,}000 = 10{,}000$.

The bottom row of sketches suggests that:

$$10 \times 10 \times 10 = 1{,}000$$
$$10 \times 10 = 100$$
$$1 \times 10 = 10$$
$$1 \times 1 = 1$$

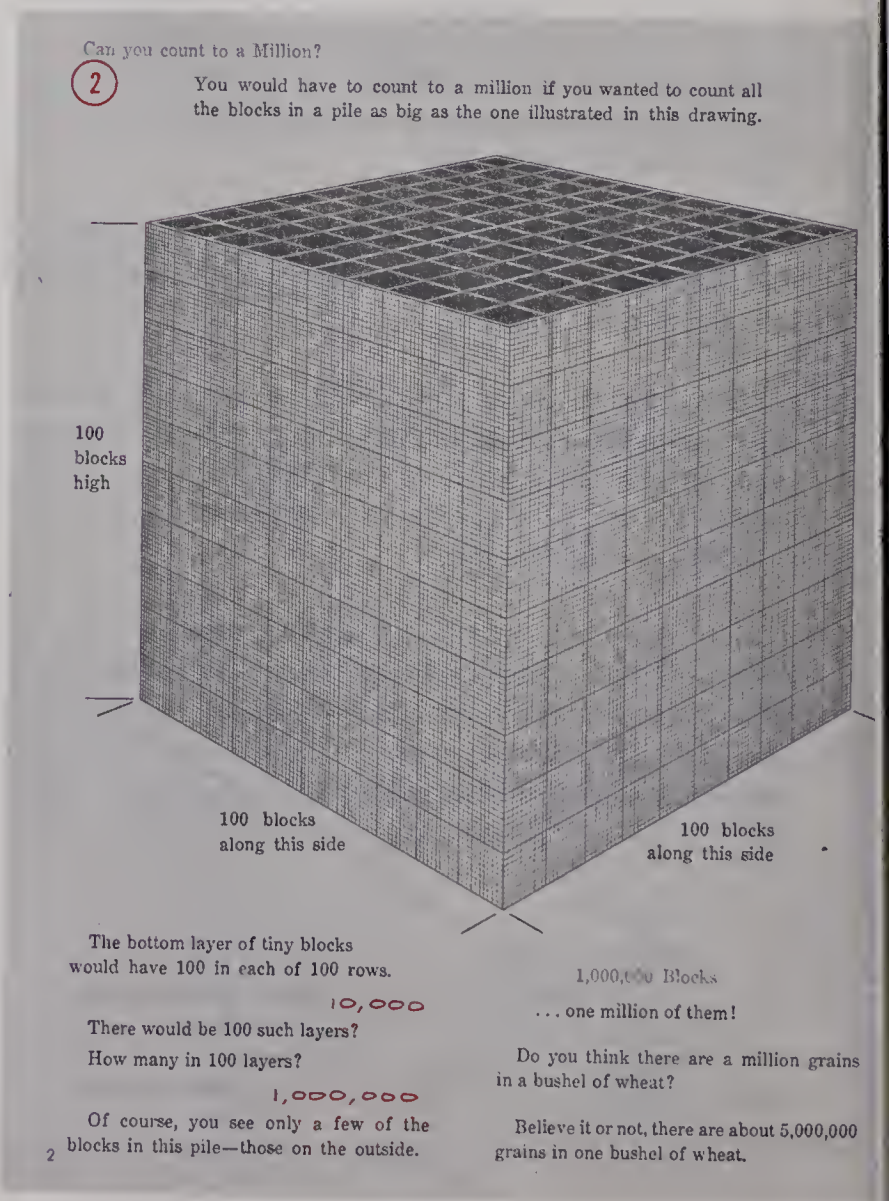
Pages 2 and 3 are intended to emphasize that 1,000,000 blocks is an awful stack—but finite. It is composed of a large collection of individual blocks.

If each sketch were drawn as a helter-skelter pile, there would be no way of knowing the actual number unless we could count them, one by one, or organize them in groups of 10, 100, 1000, etc.

"How many of the blocks got into each sketch on page 3?" Of the 100,000, we can see all the top layer—10,000. On each side, there are 1,000, but we have already counted some that are also in the top layer.

This question will lead to some arguments and, you can be sure, some independent work by some of your pupils. The correct answers are:

Sketch of:		Blocks that can be seen in picture:
100,000	→	11,791
10,000	→	1,981
1,000	→	271
100	→	100
10	→	10
1	→	1


Can you count to a Million?

2

You would have to count to a million if you wanted to count all the blocks in a pile as big as the one illustrated in this drawing.

100 blocks high

100 blocks along this side

100 blocks along this side

The bottom layer of tiny blocks would have 100 in each of 100 rows.

10,000

There would be 100 such layers?

How many in 100 layers?

1,000,000

Of course, you see only a few of the blocks in this pile—those on the outside.

1,000,000 Blocks

. . . one million of them!

Do you think there are a million grains in a bushel of wheat?

Believe it or not, there are about 5,000,000 grains in one bushel of wheat.

An approach one might take is to determine how many can't be seen:

Sketch of:	Blocks that can't be seen in picture:
100,000	→ 99 × 99 × 9 = 88,209
10,000	→ 99 × 9 × 9 = 8,019
1,000	→ 9 × 9 × 9 = 729

Work Text page 4 helps reinforce the immensity, yet finiteness, of 1,000,000 blocks each of which was only 1 cubic inch in size—1 inch by 1 inch by 1 inch.

Placed neatly, side by side, they would reach a little less than 16 miles—and closer to 15.78 miles (but a trifle more).

—that is 12 to the foot
—or 12 × 5280 per mile—63,360 per mile
—1,000,000 ÷ 63,360 = 15.78+

Then we consider the total weight if each weighed an ounce.

In considering the answer to D, we are confronted with a problem. How long is the diagonal of a 1-inch square? At this point, measure with a ruler. It is a little more than 1 3/8 inches and less than 1 7/16 inches (or, a little more than 1.414 inches). Let each child find his own approximation and use it in making computations.

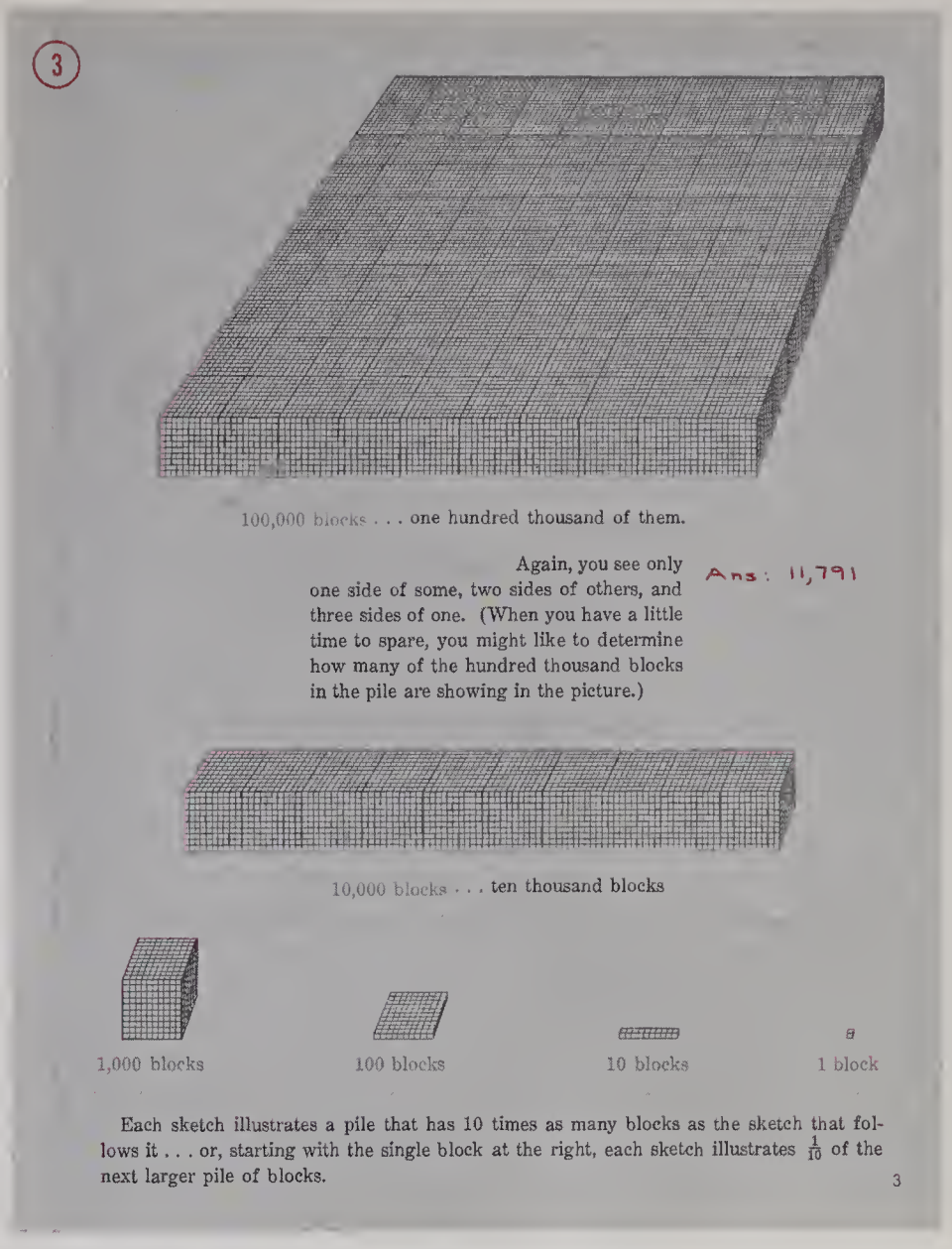
The computations on the right are just plain hard work. You may decide it's advisable to assign a limited number to each child or group of children. Some may be used on the board as refreshing examples to some who have lost some facility in the division algorism.

Work Text page 5 begins by considering that same "one cubic inch"—but not a million of them—only a single block. "If we agree that its weight is 1 ounce, what is the weight of a slice off one side that is 1/10 of an inch thick?" (.1 of an ounce.)

Next, we take a tenth of this slice—and it would weight 1/100 of an ounce, or .01 ounce. And a tenth of that would weight only .001 ounce.

"Now suppose we have fine instruments and work under a powerful magnifying glass. Can we keep up this pattern until we have a piece that weighs a millionth of an ounce?" Answer: Yes, and why stop there?

If our picture of a pile of blocks on *Work Text* page 2 was really our 1-ounce cube magnified, then



3

100,000 blocks . . . one hundred thousand of them.

Again, you see only one side of some, two sides of others, and three sides of one. (When you have a little time to spare, you might like to determine how many of the hundred thousand blocks in the pile are showing in the picture.)

Ans: 11,791

10,000 blocks . . . ten thousand blocks

1,000 blocks

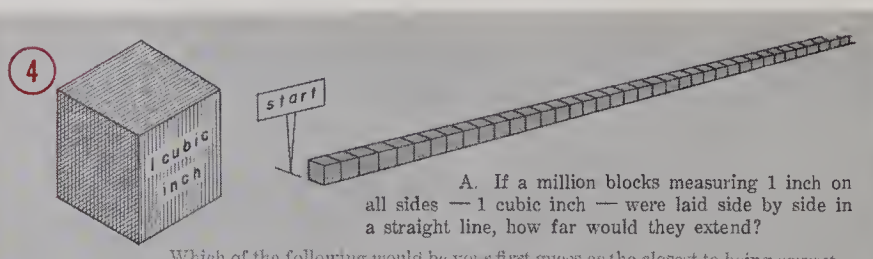
100 blocks

10 blocks

1 block

Each sketch illustrates a pile that has 10 times as many blocks as the sketch that follows it . . . or, starting with the single block at the right, each sketch illustrates $\frac{1}{10}$ of the next larger pile of blocks.

3



4

A. If a million blocks measuring 1 inch on all sides — 1 cubic inch — were laid side by side in a straight line, how far would they extend?

Which of the following would be your first guess as the closest to being correct: 500 feet; 5 miles; 15 miles; 100 miles; 770 miles.

B. There are 5,280 feet in a mile. Can you find out which distance given above is close to the correct answer?

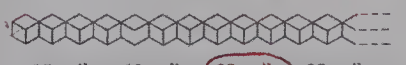
15.78+ miles

C. If each one cubic inch block of a certain material weighs 1 ounce, how many tons would a million of those blocks weigh? (There are 16 ounces in a pound and there are 2,000 pounds in a ton.) 31 $\frac{1}{4}$ tons

Which of the weights below is closest to being the correct weight?

1 ton; 19 $\frac{1}{2}$ tons; 31 $\frac{1}{4}$ tons; 87 tons

D. If a million one cubic inch blocks were laid in the pattern below, how far would they reach? 22.3+ miles



15 miles; 19 miles; 22 miles; 25 miles; 30 miles

E. How many running feet of 2" × 4" blocks can be cut from 1,000,000 cubic inches of wood?

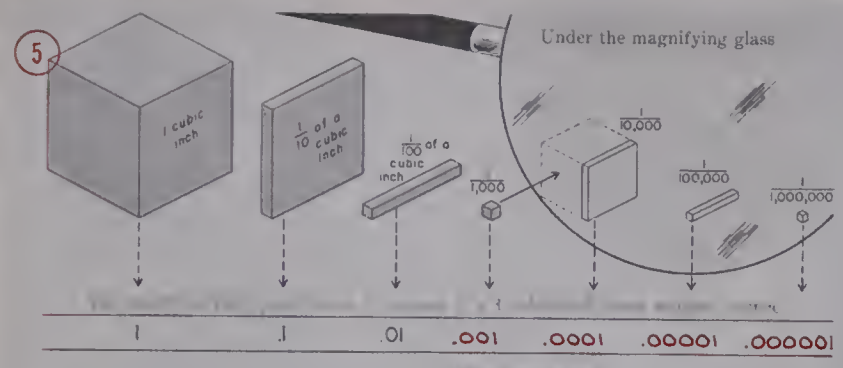
10,416 $\frac{2}{3}$ feet or almost 2 miles

F. How many square feet could be covered by 1,000,000 cubes of wood that are 1" by 1" by 1"?

6,944 $\frac{4}{9}$ square feet

1,000,000 ÷ 2 =	500,000
1,000,000 ÷ 3 =	333,333 $\frac{1}{3}$
1,000,000 ÷ 4 =	250,000
1,000,000 ÷ 5 =	200,000
1,000,000 ÷ 6 =	166,666 $\frac{2}{3}$
1,000,000 ÷ 7 =	142,857 $\frac{1}{7}$
1,000,000 ÷ 8 =	125,000
1,000,000 ÷ 9 =	111,111 $\frac{1}{9}$
1,000,000 ÷ 10 =	100,000
1,000,000 ÷ 11 =	90,909 $\frac{1}{11}$
1,000,000 ÷ 12 =	83,333 $\frac{1}{3}$
1,000,000 ÷ 13 =	76,923 $\frac{1}{13}$
1,000,000 ÷ 14 =	71,428 $\frac{4}{7}$
1,000,000 ÷ 15 =	66,666 $\frac{2}{3}$
1,000,000 ÷ 20 =	50,000
1,000,000 ÷ 25 =	40,000
1,000,000 ÷ 75 =	13,333 $\frac{1}{3}$
1,000,000 ÷ 100 =	10,000
1,000,000 ÷ 500 =	2,000
1,000,000 ÷ 750 =	1,333 $\frac{1}{3}$
1,000,000 ÷ 1,000 =	1,000

4



Does one-millionth of a cubic inch seem very small? Each side would measure .01 of an inch. Do you think you could see it without a magnifying glass? Yes

It might look like a grain of sand. But there are things much smaller. If this tiny piece one-millionth of a cubic inch were hollow like a box, it could hold many bacteria — about 15 million of them. Molecules are much smaller than bacteria. Atoms are still smaller. Electrons are still smaller —

In arithmetic, "of" and "times" indicate the same idea. And we have several ways to write fractions. Complete the following:

$\frac{1}{10}$ of 1 = $\frac{1}{10}$ .1 of 1 = .1

$\frac{1}{10}$ of $\frac{1}{10}$ = $\frac{1}{100}$ .1 of .1 = .01

$\frac{1}{10}$ of $\frac{1}{100}$ = $\frac{1}{1,000}$ .1 x .01 = .001

$\frac{1}{100}$ x $\frac{1}{100}$ = $\frac{1}{10,000}$ .01 x .01 = .0001

$\frac{1}{10}$ x $\frac{1}{1,000}$ = $\frac{1}{10,000}$ .1 x .001 = .0001

$\frac{2}{10}$ x $\frac{2}{10}$ = $\frac{4}{100}$ .2 x .2 = .04

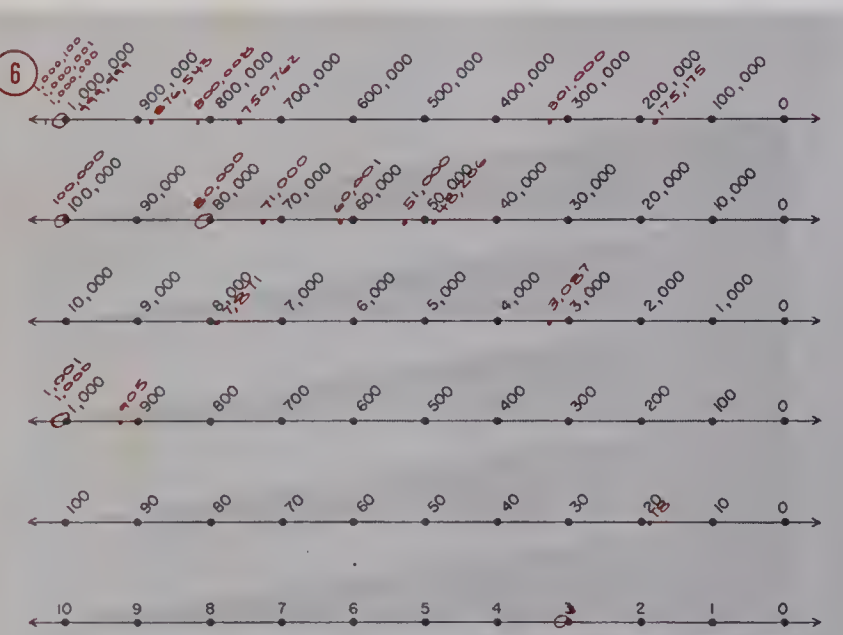
$\frac{2}{10}$ x $\frac{6}{10}$ = $\frac{12}{100}$ .2 x .6 = .12

$\frac{5}{100}$ x $\frac{5}{100}$ = $\frac{25}{10,000}$ .05 x .05 = .0025

$\frac{3}{10}$ x $\frac{15}{1,000}$ = $\frac{45}{10,000}$ .3 x .015 = .0045

.3 + .7	.1 x 10	7 ÷ .7
1.4 - .4	.19 ÷ .19	.92 + .08
0 + 1.0	1.012 - .012	.25 x 4
.2 x 5	.07 + .93	10 x .1
8 - 7	100 x .01	2.3 - 1.3
100 x .01	.81 + .19	93 ÷ 93
2.81 - 1.81	2 x .5	.16 + .84
40 x .025	.007 + .993	.02 x 50
.2 ÷ 2	.04 + .06	1.4 - 1.3
1 x .1	.2 x .5	1 ÷ 10
100 x .001	.01 x 10	.011 + .089
.01 + 0	.01 x 1	.8 - .79
.001 + .009	10 x .001	5 x .002
.133 - .123	.009 + .001	4 x .0025
2 x .005	.0099 + .0001	100 x .0001

5



It's easy to read large numbers:

79 , 347 , 205 , 870

Billions Millions Thousands

Start reading the numbers at the left, and use the indicated words as you come to each comma:

"Seventy nine , three hundred forty-seven , two hundred five , eight hundred seventy."

. . . unless there are 3 zeros to the left of a comma. Then, don't say anything when you pass the 3 zeros and the comma that follows it.

2 , 120 , 000 , 061

"Two , one hundred twenty , sixty-one."

Read the following numbers. Then locate them on a number line. Unless they are starred (*).

3	3,087
18	750,762
905	8,400,036*
1,000	1,001,358,265*
1,001	71,000
80,000	8,000,762*
48,206	127,138,000*
175,175	1,000,001
301,000	100,000,984*
800,008	60,001
51,000	100,000
7,871	876,543
999,999	1,000,100
1,000,000	1,700,000

6

each tiny block would be a magnified picture of a block that would weight .000001 of an ounce.

The rest of the page is refresher work in multiplication of decimals.

Work Text page 6 takes another look at "one million." "Consider 1,000,000 as a point on a number line. What would we label the point half way between 0 and 1,000,000? . . .one-tenth of the way from 0 to 1,000,000? . . .a fifth of the way? . . .a fourth of the way?"

The second number line on page 6 is a magnification of that section of the line above that reaches one-tenth of the way from 0 to 1,000,000—from 0 to 100,000. Each number is intended as a magnification of a section of the number line above.

If any children still have trouble reading large numbers, develop (as extensively as required) the helpful technique shown on *Work Text* page 6. Put the following on the board:

A B C D

79,347,205,870

"billion" "million" "thousand"

"Read each group of numbers set off by commas and labeled by letters:

A → seventy-nine
B → three hundred forty-seven
C → two hundred five
D → eight hundred seventy

Change the numerals, and repeat—until everyone understands.

"Now, read each group from left to right, but *pronounce* each comma as it is labeled: 79 billion 347 million 205 thousand 870." That's all there is to it with one exception: no comma is "pronounced" if it is preceded by a string of 3 zeros. Thus:

1,000,000

million

is read "one million"—and that's all.

187,000,017,000

is read "187 million 17 thousand."

Urge children *not* to insert "and" in reading such numbers as 105: "one hundred five," not "one hundred *and* five," because we will use this in writing decimal fractions on *Work Text* page 7.

When a decimal point is indicated, pronounce it "and" unless there is no numeral 1 through 9 written on the left:

72 , 008 , 010.7

"million" "thousand" "and"

Read: "72 million 8 thousand 10 and 7-tenths."
0.7
Read, "7-tenths."

By pointing to this simple pattern that some have nicknamed "pronouncing the punctuation marks," the reading difficulties disappear.

Some children inevitably want to know how successive commas are read to the left of billions. The list begins:

1,000 units	—1 thousand
1,000 thousand	—1 million
1,000 million	—1 billion
1,000 billion	—1 trillion
1,000 trillion	—1 quadrillion
1,000 quadrillion	—1 quintillion
1,000 quintillion	—1 sextillion
1,000 sextillion	—1 septillion

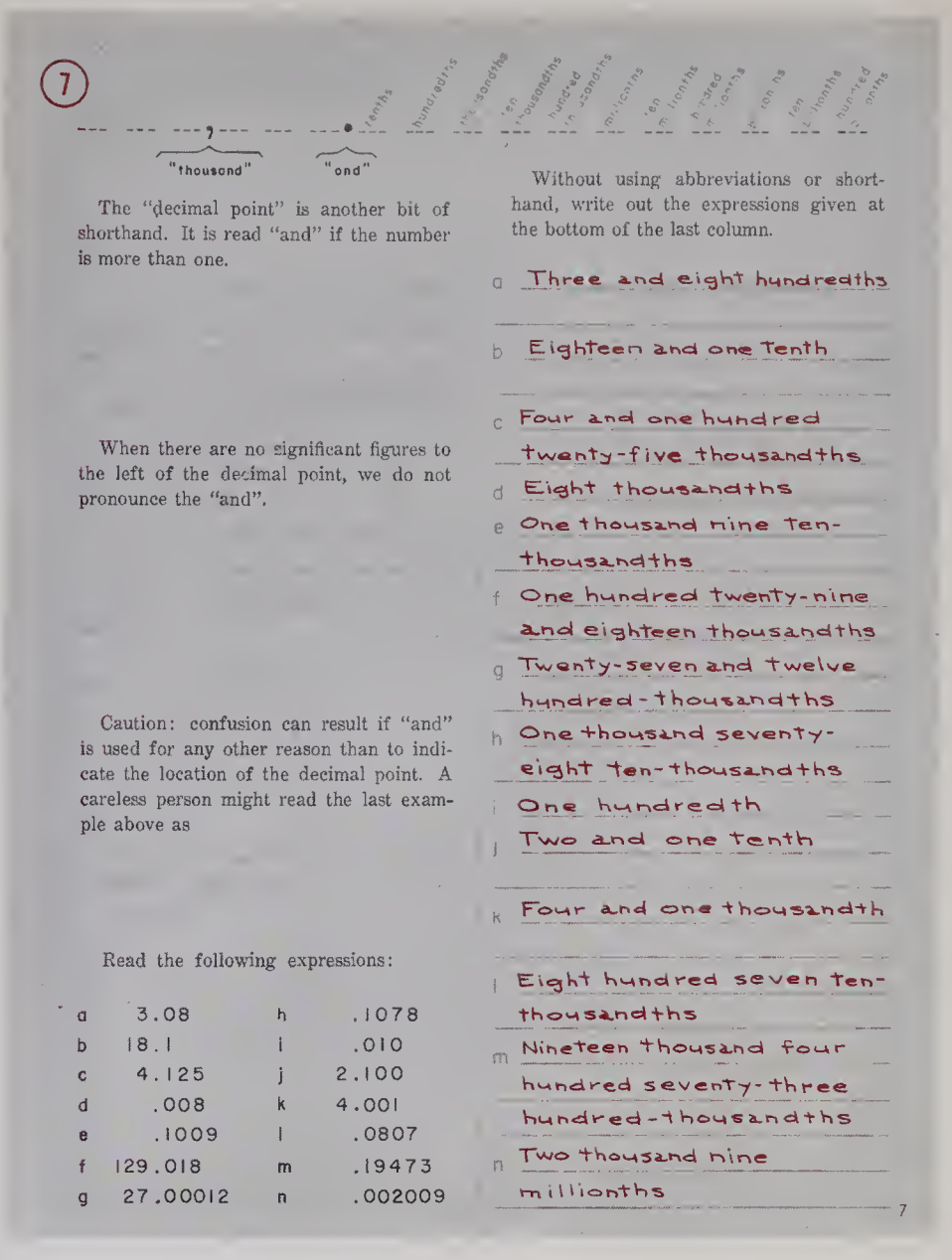
and that is written:

1,000,000,000,000,000,000,000,000
and read, "One septillion."

(The list goes on—but no one uses these expressions. We will consider an alternate method later—the method scientists use.)

Work Text pages 2 through 7 range back and forth between very large and very small numbers and measurements and things. Our main concern has been with what we call our decimal system of numeration. It is a pattern that we extend as far as we need to talk about measurements of interstellar distance or the weight of an electron.

It has been a hurried trip we have taken, but our purpose is to get an overview of a great and general pattern we will keep studying for different reasons and at different times throughout *Level F*.



7

"thousand" "ond"

The "decimal point" is another bit of shorthand. It is read "and" if the number is more than one.

When there are no significant figures to the left of the decimal point, we do not pronounce the "and".

Caution: confusion can result if "and" is used for any other reason than to indicate the location of the decimal point. A careless person might read the last example above as

Read the following expressions:

a	3.08	h	.1078
b	18.1	i	.010
c	4.125	j	2.100
d	.008	k	4.001
e	.1009	l	.0807
f	129.018	m	.19473
g	27.00012	n	.002009

Without using abbreviations or shorthand, write out the expressions given at the bottom of the last column.

a Three and eight hundredths

b Eighteen and one Tenth

c Four and one hundred twenty-five thousandths

d Eight thousandths

e One thousand nine ten-thousandths

f One hundred twenty-nine and eighteen thousandths

g Twenty-seven and twelve hundred-thousandths

h One thousand seventy-eight ten-thousandths

i One hundredth

j Two and one tenth

k Four and one thousandth

l Eight hundred seven ten-thousandths

m Nineteen thousand four hundred seventy-three hundred-thousandths

n Two thousand nine millionths

7

Notes to accompany *Work Text* pages 8-11

"What's My Rule?" is a fascinating and enlightening game for children to play. Start off, one day, by asking a child to call out a number.

"Six."

"I answer by saying twelve. Someone else call out any number."

"Three and one-half."

"I answer by saying 7."

"170."

"340," you answer. "How many know my rule?" Several want to tell you—but you will only permit this kind of revealing statement:

"If I call out 8," Mary explains, "you will call out 16."

"That's right." Others volunteer. A few don't understand, so record the results: ("six goes to twelve")

8

When Mr. Dean walked into the arithmetic class one day, he saw the chart below on the board — the part shown in black. Jenny was calling out numbers.

"Four," she said, and wrote 4 at the top of the third column from the left.

The others wrote numbers opposite their names in the same column.

"I've got the rules," Henry spoke up. "After Mary's name and under 10, I would write 17. In the same column, I would write 20 after Bill's name, 9 after Alice's name, 21 after George's name, and 30 after Al's name."

Numbers Jennie called out	3	1	[illegible]	[illegible]	[illegible]	Rules:	Shorthand n
Mary	10	8	[illegible]	17	16		$n+7$
Bill	6	2	[illegible]	20	18		$2n$
Alice	2	0	[illegible]	9	8		$n-1$
George	7	3	[illegible]	21	19		$2n+1$
Al	9	3	[illegible]	30	27		$3n$

(Complete the record in the fourth and fifth columns.)

Because Henry had discovered the rules in the game above, he took Jennie's place and called out the numbers. He selected ten members of the class to make up secret rules. Can you discover the rules from the records given? Can you find a shorthand expression for the rules?

Numbers called out	2	9	25	7	13	Rules: (Note: There are many ways to state each rule in words)	Shorthand a
Joe	4	[illegible]	50	14	26	Double the number called out.	$2a$
Bob	0	[illegible]	23	5	11	Subtract two from the number called out.	$a-2$
Catherine	5	19	[illegible]	5	27	Double the number called out and then add one.	$2a+1$
Sandra	13	20	36	18	24	Add eleven to the number called out.	$a+11$
Sue	3	17	49	13	25	Double the number called out and then subtract one	$2a-1$
Bill	6	27	75	21	39	Triple the number called out: or multiply it by three.	$3a$
Dorothy	7	21	53	17	29	Double the number called out and add three.	$2a+3$
June	9	37	101	29	53	Add one to four times the number called out.	$4a+1$
Ed	4	25	73	19	37	Triple the number called out and then subtract two.	$3a-2$
Helen	11	46	126	36	66	Multiply the number called out by five and then add one.	$5a+1$

8

9

I

3	8
0	5
10	15
5	10
7	12
9	14
n	$n+5$
20	25
81	86
7.5	12.5
$\frac{1}{2}$	$5\frac{1}{2}$

II

7	14
23	46
0	0
1	2
2	4
8	16
a	$2a$
19	38
27	54
4.3	8.6
$\frac{1}{3}$	$\frac{2}{3}$

III

0	0
1	3
2	6
10	30
7	21
19	57
s	$3s$
28	84
50	150

IV

19	17
86	84
2	0
7	5
21	19
50	48
r	$r-2$

V

2	5
5	11
7	15
0	1
6	13
4	9
g	$2g+1$
10	21
50	101
$\frac{1}{3}$	$1\frac{2}{3}$

VI

0	0
3	21
10	70
5	35
9	63
13	91
h	$7h$
28	196
44	308
13	91
$\frac{1}{2}$	$3\frac{1}{2}$

VII

1	12
19	30
11	22
7	18
8	19
26	37
t	$t+11$
49	60
$1\frac{1}{2}$	$12\frac{1}{2}$
89	100

VIII

3	16
7	24
0	10
25	60
9	28
17	44
x	$2x+10$
$\frac{1}{2}$	11
$2\frac{1}{2}$	15
$\frac{1}{4}$	$10\frac{1}{2}$

IX

20	0
5	15
17	3
10	10
12	8
19	1
y	$20-y$
0	20
1	19
$12\frac{1}{2}$	$7\frac{1}{2}$
$14\frac{1}{3}$	$5\frac{2}{3}$

X

10	107
0	7
5	57
$\frac{1}{2}$	12
20	207
2	27
z	$10z+7$
8	87
25	257
9	97
.2	9

XI

5	25
7	49
3	9
2	4
8	64
11	121
n	$n \cdot n$
10	100

XII

5	20
7	42
3	6
1	0
10	90
9	72
b	$b \cdot b - b$
11	110
20	380
15	210
9	72

XIII

14	7
9	$4\frac{1}{2}$
50	25
4	2
28	14
31	$15\frac{1}{2}$
x	$x \div 2$
58	29
174	87
190	95
1	$\frac{1}{2}$

XIV

1	.7
4	2.8
.1	.07
20	14
10	7
100	70
y	$.7y$
50	35
or $\frac{x}{2}$	

XV

3	5
10	19
7	13
50	99
5	9
9	17
n	$2n-1$
59	117
or $\frac{7}{10}y$	or $\frac{7y}{10}$

9

As the blanks are filled in, all members of the class will be quite sure they know your rule. "You know my rule because you saw a pattern in the examples. Which of the following expressions would you select as an appropriate name for my rule—for the pattern you noticed?"

I. $c \rightarrow 3r$	IV. $x \rightarrow x + x$
II. $b \rightarrow 2 \times b$	V. $q \rightarrow 2q$
III. $m \rightarrow 2 \times s$	VI. $d \rightarrow e + f$

Candidates will include:

$b \rightarrow 2 \times b$
$x \rightarrow x + x$
$q \rightarrow 2q$

Each seems to have its merits, but do not let the discussion become involved. You might make the comment that the third candidate is included because "2q" can be a shorter way of writing "$2 \times q$" and "$q \times 2$." The purpose is simply to introduce in a general way the idea of selecting a name for a pattern.

"New game! New rule! Someone call out a number."

"5."

"11," you answer.

"10."

"21," you answer.

A record of the game might be:

$5 \rightarrow 11$	$8 \rightarrow 17$
$10 \rightarrow 21$	$\frac{1}{2} \rightarrow 2$
$7 \rightarrow 15$	$100 \rightarrow 201$
$0 \rightarrow 1$	$6 \rightarrow 13$
$3 \rightarrow 7$	$2 \rightarrow 5$

"If I say 6, you will say 13." Right! That's the legal way for a player to indicate that he has discovered the rule.

"If I say 2, you will answer 5." Right!

Finally: "Let's select a name for the pattern you see in this group of examples. Which of the following seems to express the idea in the pattern?"

I. $p \rightarrow q + 2$	IV. $r \rightarrow r + 1 + r$
II. $n \rightarrow 2n + 1$	V. $x \rightarrow 3x - 1$
III. $a \rightarrow 3a + 1 - a$	VI. $s \rightarrow 1 + 2 \times s$

Candidates will be II, IV and, perhaps, VI. Also expect an argument for III. All, of course, could well serve as names for the pattern. Only I and V don't have much in common with the pattern.

Mention that "2n + 1" is a good name for the pattern if "2n" and "2 × n" indicate the same idea.

Relinquish the role of teacher. Let the children make up the rules. As each game becomes obvious, ask the children to suggest nice short names that seem to express the idea in the pattern. Eventually, have several children think up rules at the same time. As a number is called out, each of the rule makers answers. Keep track of developments.

Numbers called out	↓ 3	↓ 4	↓ 10	7			
Carl	9	12	30	21			$a \rightarrow 3a$
Helen	10	14	20	17			
Jack	6	8	20	14			

Someone calls out 3: Carl answers 9; Helen answers 10; Jack answers 6.

Another volunteer calls out 4: Carl answers 12; Helen answers 14: Jack answers 8.

To 10, Carl answers 30; Helen, 20; and Jack, 20.

Now Mabel thinks she know all three rules. She says, "If I say 7, Carl will say 21, Helen will say 17 and Jack will say 14."

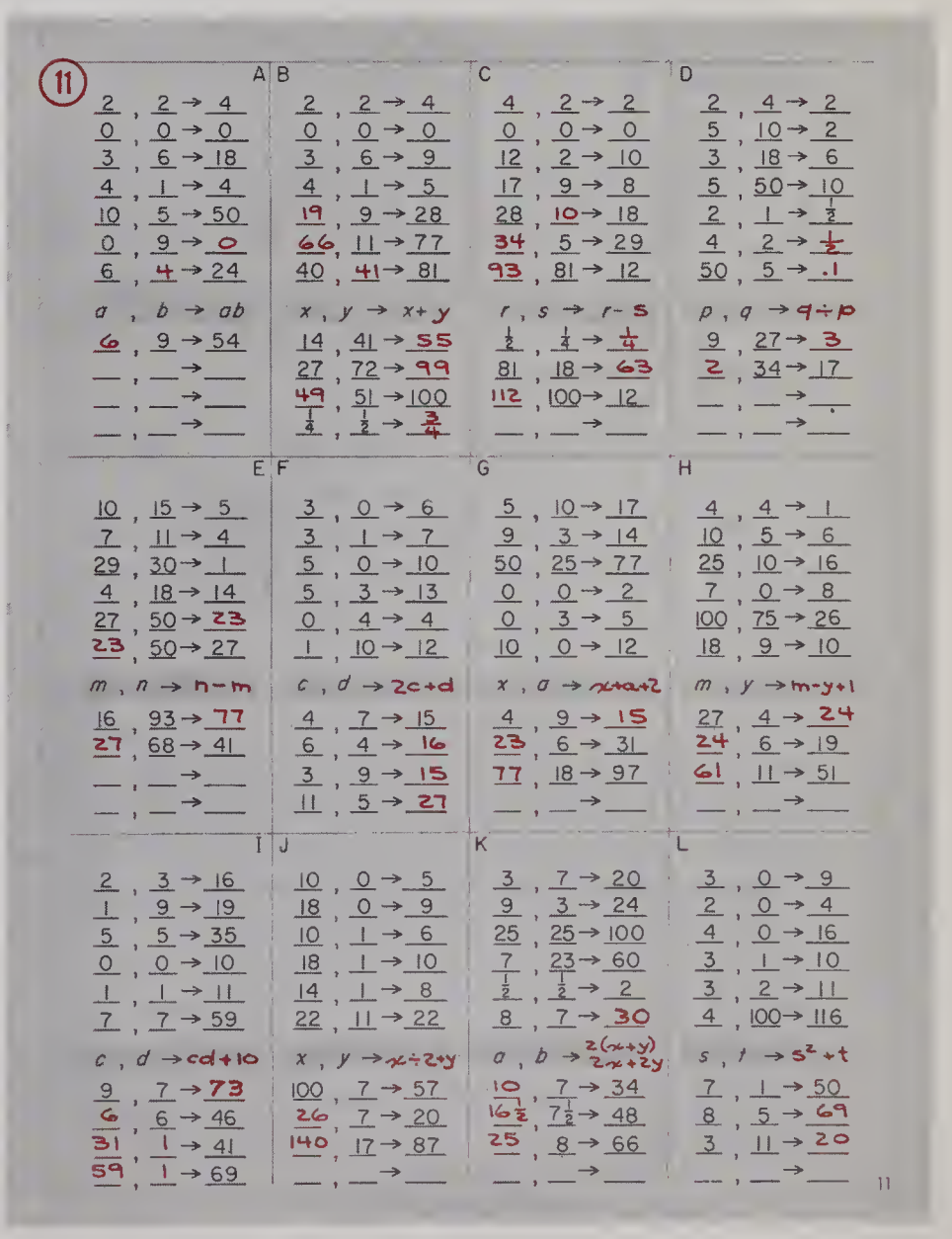
"Right," to all three answers.

As the examples are worked out, soon everyone sees the three patterns.

"Now, let's suggest names for Carl's pattern." You may start several:

a →	r →
x →	q →

Proposals may be:



$a \rightarrow 3a$ $\qquad$ $r \rightarrow 2r + r$
$x \rightarrow x + x + x$ $\qquad$ $q \rightarrow 3 \times q$

To Carl: "Which of these express the idea of your rule?" He may answer that all are appropriate.

"Which shall we write in our chart at the end of the line with your answers?" Label that row with his choice. In a similar way, Helen and Jack select names for their rules from a group of candidates.

After several such games, the class can tackle *Work Text* page 8 with little difficulty. The following three pages—9, 10, and 11—are specially constructed for independent work by each child.

Page 9 begins with very simple patterns—and all, or most, of an appropriate title given. The children complete some examples, and make others of their own. Some may not be familiar with the expressions "n · n" and "b · b" in the bottom row. Explain that a dot (high enough not to be confused with a decimal point) is sometimes used in place of a multiplication sign:

$3 \times 7 = 3 \cdot 7$
$8 \times 1 = 8 \cdot 1$
$9 \times 9 = 9 \cdot 9$

Patterns on page 9 increase in difficulty—and more of the title boxes are left for children to complete.

If problems arise, be sure that each child completes all the examples in a group before he worries about a "title." If he doesn't have one in mind when he reaches the box, have him work on making up his own examples—so you are sure he has actually grasped the pattern.

The names selected should not be characterized as "right" or "wrong." If you feel a certain selection is unfortunate, offer a group of alternatives that do express the idea and let each child select from the list.

The whole idea of mixing letters and numerals and familiar symbols of arithmetic is new to many. We do not, under any circumstances, talk about rules—about correct and incorrect usage. Let the idea form slowly. Many examples are given, and children soon see the technique we use to point out patterns. There is much more to come.

Pattern indicator

Page 10 makes a shift. Each example starts with a "pattern indicator." After 1, no illustrations of the patterns are given.

Encourage discussion of the work on this page between children in small groups of two or three. Let them argue among themselves about examples that fit the pattern names. If your help is required, limit it to providing one or two or three examples. The less talk from the teachers at this point, the better.

If a child asks: "Are those letters like numbers?" the answer ought to be indirect. "They are simply part of a name for a pattern. They help us point out the patterns we see." (You may be relieved to know that mathematicians aren't at all agreed on the answer to "What are those letters?".) It may be that some children should save some of the work for later.

Page 11 introduces an expansion of the idea. If the going is rough, try an appropriate version of "What's My Rule?"

"Call out two numbers."

"6 and 3."

"I answer 18. Call out another pair."

"2 and 5."

"I answer 10."

Keep a record (6 and 3 goes to 18):

6, 3 → 18
2, 5 → 10
3, 3 → 9
4, 7 → 28
1, 9 → 9
2, 9 → 18
6, 5 → ____
4, 10 → ____

"Now, let's select a name for the pattern.:

$a, b \rightarrow a + b$ $\qquad$ $m, n \rightarrow n \times m$
$r, s \rightarrow r\ s$ $\qquad$ $q, c \rightarrow cq$

All would be readily acceptable except the first—it doesn't seem to express the pattern we notice at all.

A series of such games with full class participation will be adequate readiness to get the page underway. Some of the patterns become quite difficult to detect. They are puzzles. Do not help much. These are "brain busters," so urge the children to save them for study later if they are too difficult now. Often, a puzzle comes much easier

when it is tackled with a fresh approach at another time. Page 11 can be finished much later.

Notes to accompany *Work Text* page 12

Page 12 is a brief refresher concerned with the fact that decimal fractions and common fractions are simply variations in the shorthand we use to express an idea.

"Let us agree that each of the diagrams below suggests the idea of 1." That kind of formulation is used repeatedly throughout the *Work Text.* It avoids such clumsy and misleading statements as: "Here are some pictures of one." One is an abstract idea. Soon we would have a whole collection of different "pictures of one"—as if *one* were a quick-change artist. Actually, all the different pictures would have a common aspect—which characteristic we have named "one." Each of the pictures or sketches could be used to *suggest* the idea—which is the case on *Work Text* page 12.

"The area of this diagram is 1." But, "1 what?" About 1 square inch. Certainly the area is not the abstract idea of 1.

Usually, in mathematics, sketches or diagrams are used to suggest an idea. Often we use several pictorial representations to suggest an idea—each as different from the other as possible. This helps focus attention away from actual details of the sketch and on that common aspect of the sketches which is the idea we are talking about.

None of us is apt to be a purist in areas such as this. However, the expression ". . . suggests the idea of . . ." can be used often enough to keep alive the distinction between an idea and any of the diagrams we use to point to it.

"That first sentence on page 12 has another implication. It says that 'each of the sketches' will be used to 'suggest the idea of 1.' What do the sketches have in common?" Many things: each has four corners; the diagonal of each is about $\sqrt{2}$ inches; etc. But they all are about the "same size." So, it is implied that the size will be used to indicate the idea of 1.

If that is true, then a portion of the area—a part that is smaller in size—will "suggest the idea" of numbers that are less than 1.

In example A, the blue-shaded area is intended to suggest the idea of 1/2 or .5; the green-shaded area is intended to suggest the idea of 1/10 or.1; the un-shaded, 4/10 or .4; the blue and green together, 6/10 or 3/5 or .6; etc.

"Should 6/10 always be reduced to 3/5?" is a question often asked by teachers and children and parents. It would certainly be inadvisable in the following expression to change 6/10 to 3/5:

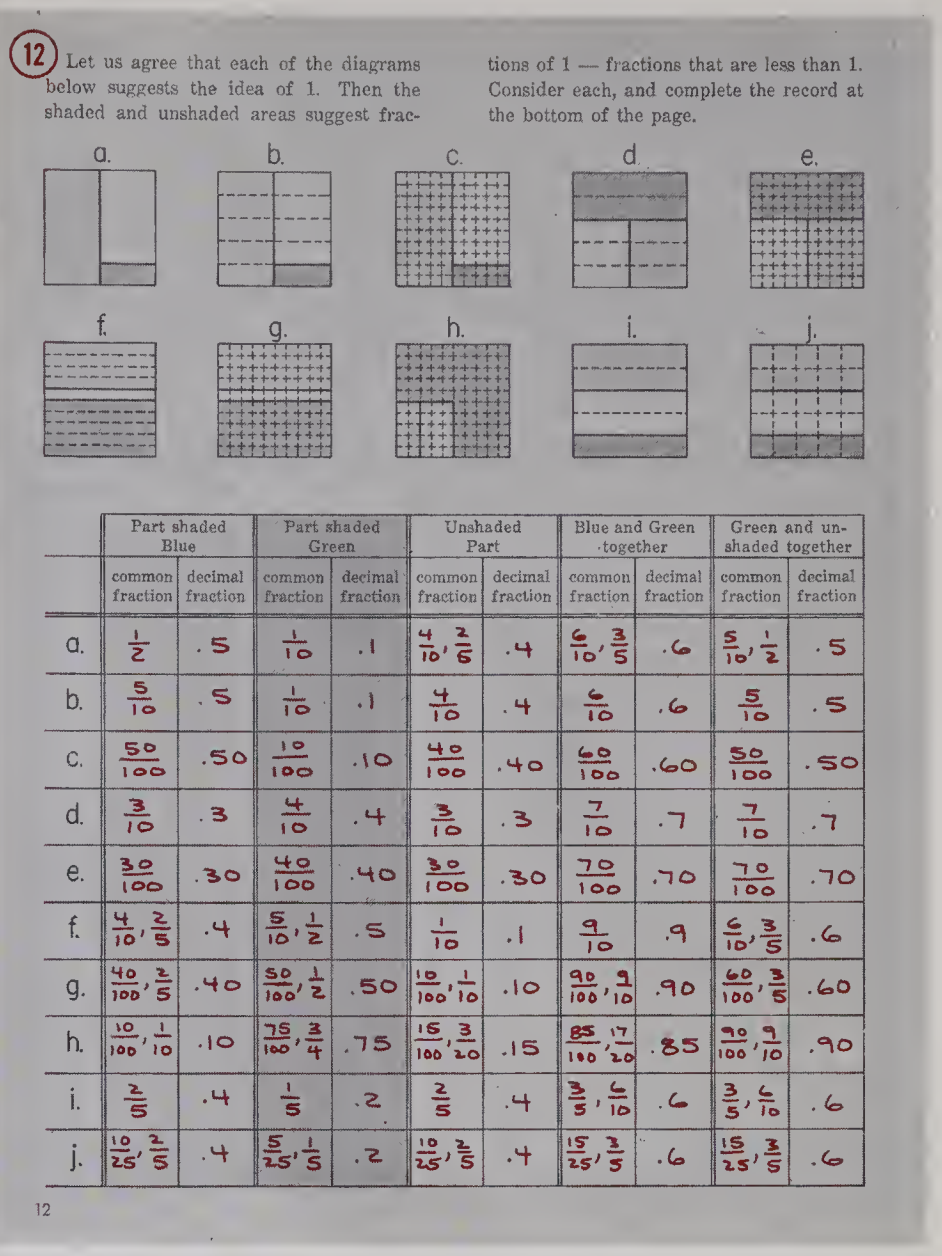
$$6/10 + 3/10 = \_\_\_\_$$

However, it is the custom to carry out computation a step further in most cases, such as:

$$1/10 + 5/10 = \underline{6/10}$$

and some standardized achievement tests would score 6/10 as an improper conclusion in the example above.

This situation must be explained to children. You might even tell them that, to keep them on the alert, you are going to pass a special law, for a specified period of time, that: When the conclusion of a mathematical statement contains a fraction with a common factor in the numerator and denominator, the greatest common factor should be pointed out by "changing the expression to a fraction with no common factor other than one in the numerator and denominator." But point out that this is a special



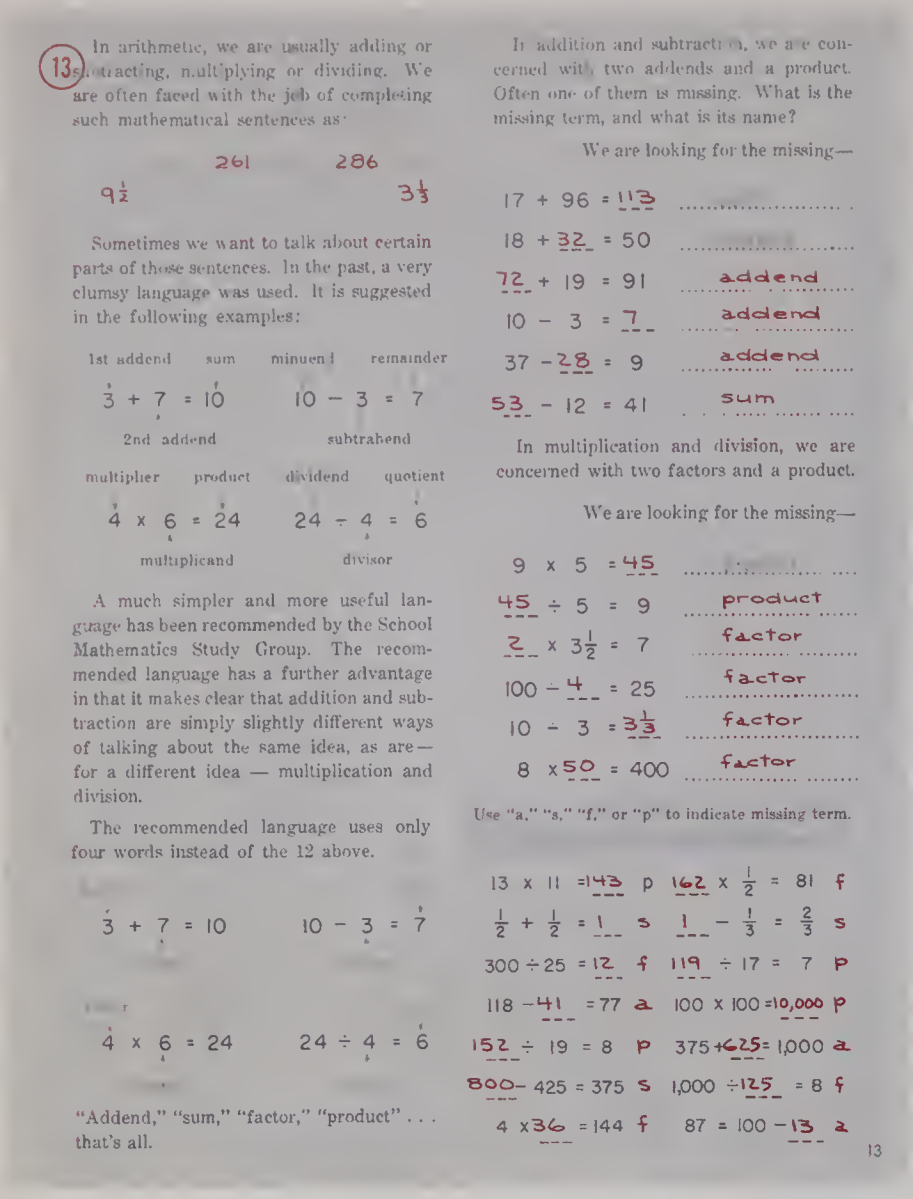
(12) Let us agree that each of the diagrams below suggests the idea of 1. Then the shaded and unshaded areas suggest fractions of 1 — fractions that are less than 1. Consider each, and complete the record at the bottom of the page.

a. b. c. d. e.

f. g. h. i. j.

	Part shaded Blue		Part shaded Green		Unshaded Part		Blue and Green together		Green and unshaded together	
	common fraction	decimal fraction	common fraction	decimal fraction	common fraction	decimal fraction	common fraction	decimal fraction	common fraction	decimal fraction
a.	$\frac{1}{2}$	.5	$\frac{1}{10}$	.1	$\frac{4}{10}, \frac{2}{5}$	.4	$\frac{6}{10}, \frac{3}{5}$	.6	$\frac{5}{10}, \frac{1}{2}$	.5
b.	$\frac{5}{10}$	.5	$\frac{1}{10}$	.1	$\frac{4}{10}$	.4	$\frac{6}{10}$	.6	$\frac{5}{10}$	.5
c.	$\frac{50}{100}$	.50	$\frac{10}{100}$	.10	$\frac{40}{100}$	.40	$\frac{60}{100}$	.60	$\frac{50}{100}$	.50
d.	$\frac{3}{10}$	.3	$\frac{4}{10}$	.4	$\frac{3}{10}$	.3	$\frac{7}{10}$	.7	$\frac{7}{10}$	.7
e.	$\frac{30}{100}$	.30	$\frac{40}{100}$	.40	$\frac{30}{100}$	.30	$\frac{70}{100}$	.70	$\frac{70}{100}$	.70
f.	$\frac{4}{10}, \frac{2}{5}$	.4	$\frac{5}{10}, \frac{1}{2}$	.5	$\frac{1}{10}$	.1	$\frac{9}{10}$	.9	$\frac{6}{10}, \frac{3}{5}$	.6
g.	$\frac{40}{100}, \frac{2}{5}$	.40	$\frac{50}{100}, \frac{1}{2}$	.50	$\frac{10}{100}, \frac{1}{10}$	.10	$\frac{90}{100}, \frac{9}{10}$	.90	$\frac{60}{100}, \frac{3}{5}$	.60
h.	$\frac{10}{100}, \frac{1}{10}$	.10	$\frac{75}{100}, \frac{3}{4}$	.75	$\frac{15}{100}, \frac{3}{20}$	.15	$\frac{85}{100}, \frac{17}{20}$	.85	$\frac{90}{100}, \frac{9}{10}$	.90
i.	$\frac{2}{5}$	.4	$\frac{1}{5}$	.2	$\frac{2}{5}$	.4	$\frac{3}{5}, \frac{6}{10}$	.6	$\frac{3}{5}, \frac{6}{10}$	.6
j.	$\frac{10}{25}, \frac{2}{5}$	.4	$\frac{5}{25}, \frac{1}{5}$	.2	$\frac{10}{25}, \frac{2}{5}$	.4	$\frac{15}{25}, \frac{3}{5}$	.6	$\frac{15}{25}, \frac{3}{5}$	.6

12



13

In arithmetic, we are usually adding or subtracting, multiplying or dividing. We are often faced with the job of completing such mathematical sentences as:

261 286

$9\frac{1}{2}$ $3\frac{1}{3}$

Sometimes we want to talk about certain parts of those sentences. In the past, a very clumsy language was used. It is suggested in the following examples:

1st addend sum minuend remainder

3 + 7 = 10 10 − 3 = 7

2nd addend subtrahend

multiplier product dividend quotient

4 x 6 = 24 24 ÷ 4 = 6

multiplicand divisor

A much simpler and more useful language has been recommended by the School Mathematics Study Group. The recommended language has a further advantage in that it makes clear that addition and subtraction are simply slightly different ways of talking about the same idea, as are—for a different idea — multiplication and division.

The recommended language uses only four words instead of the 12 above.

3 + 7 = 10 10 − 3 = 7

4 x 6 = 24 24 ÷ 4 = 6

"Addend," "sum," "factor," "product" . . . that's all.

In addition and subtraction, we are concerned with two addends and a product. Often one of them is missing. What is the missing term, and what is its name?

We are looking for the missing—

17 + 96 = 113 .....

18 + 32 = 50 .....

72 + 19 = 91 addend

10 − 3 = 7 addend

37 − 28 = 9 addend

53 − 12 = 41 sum

In multiplication and division, we are concerned with two factors and a product.

We are looking for the missing—

9 x 5 = 45 .....

45 ÷ 5 = 9 product

2 x $3\frac{1}{2}$ = 7 factor

100 ÷ 4 = 25 factor

10 ÷ 3 = $3\frac{1}{3}$ factor

8 x 50 = 400 factor

Use "a," "s," "f," or "p" to indicate missing term.

13 x 11 = 143 p 162 x $\frac{1}{2}$ = 81 f

$\frac{1}{2} + \frac{1}{2} = 1$ s $1 - \frac{1}{3} = \frac{2}{3}$ s

300 ÷ 25 = 12 f 119 ÷ 17 = 7 p

118 − 41 = 77 a 100 x 100 = 10,000 p

152 ÷ 19 = 8 p 375 + 625 = 1,000 a

800 − 425 = 375 s 1,000 ÷ 125 = 8 f

4 x 36 = 144 f 87 = 100 − 13 a

13

ruling you are making to keep the children alert for tests and common usage.

Also, you must be flexible in policing your rule. Suppose this question is asked about example *a* on page 12: "What part of the area is shaded blue?" The answers are given as follows: "Half," Mary answers. "Five-tenths," Bill replies. "Fifty per cent," Jerry answers.

If you tell Bill to "eliminate the common factor," he may ask what that means in the case of .5. If you extend the rule to require that Bill accept 1/2 as a better conclusion than .5, then what do you do with Jerry?

There are rules that are dictated by nature—and the basic rules of arithmetic reflect the observations of nature. In addition to these basic rules, there are ordinances passed that vary from community to community. If you violate those ordinances, you may be reprimanded or fined by a lower score. If you violate the basic rules, you are in much more serious trouble.

We ought to reduce the number of local ordinances to a bare minimum and thus provide the fewest diversions from the basic rules of mathematics. "Reduction of fractions" is an ordinance of doubtful value and open to much misinterpretation—but it is still on the books in some communities.

Notes to accompany *Work Text* page 13

A contrast is drawn on page 13 between an almost outmoded body of language and a more convenient and helpful handful of terms.

Mathematicians search for simplicity and generality. They are continually saying, "We seem to be confronted by two different ideas. However, if you look at it in a different way, you will see that each is simply a special case of a single principle."

One very important idea that helps unify is to think of addition as a basic idea, and subtraction as its inverse; to think of multiplication as a basic idea, and division as its inverse. Another is the recognition of the two patterns indicated below:

$$\begin{array}{ll} 3 + 7 = 7 + 3 & 2 \times 8 = 8 \times 2 \\ 9 + 2 = 2 + 9 & 4 \times 3 = 3 \times 4 \\ 4 + 5 = 5 + 4 & 10 \times 8 = 8 \times 10 \\ a + b = b + a & a \times b = b \times a \end{array}$$

When we are confronted with a distinction between "multiplier" and "multiplicand," the following basic idea is confusing (when it ought to be crystal clear): $2 \times 8 = 8 \times 2$.

What do we mean by asking, "Which is the multiplier and which is the multiplicand?" Does the expression argue that "the multiplier became the multiplicand?"

Then the mathematician comes to our rescue by pointing out that the distinction is without importance, without meaning. Further, the effort to make the distinction directs attention to trivia, when we ought to focus on the idea that:

$$\begin{array}{l} 3 \times 7 = 7 \times 3 \\ 8 \times 5 = 5 \times 8 \\ m \cdot n = n \cdot m, \end{array}$$

an idea mathematicians call "the commutative principle" of multiplication.

They recommend that we look at the indicated multiplication as a situation in which a pair of factors point to a product—and that the order in which we consider the pair is immaterial. Where two factors are given, we can find the product. If the product and a factor are given, we can find the other factor. In other words, we can complete the following sentences:

$3 \times 5 =$ ____
$7 \times$ ____ $= 42$
____ $\times 8 = 24$

They continue arguing that division is but another way of recording these same ideas:

____ $\div 3 = 5$
$42 \div$ ____ $= 7$
$24 \div 8 =$ ____

In either case, we note that a term is missing—either a factor or a product. We proceed to find the missing term.

A similar analysis of addition and subtraction points to the fact that in both, a pair of addends and a sum is involved. If a term is missing—one of the addends or the sum—we can find it.

In the "old language" we learned many names for parts of expressions as if they were details of different ideas. In the "new language" (really quite ancient) we need only four basic terms to describe the simple patterns involved.

When children are first faced with the expression: $3/4 \div 1/8 =$ ____, it is puzzling. If, however, they see it as an example of a missing factor they know it can be rewritten as:

____ $\times 1/8 = 3/4$ or $1/8 \times$ ____ $= 3/4$

And this, in the beginning, is much more familiar.

Each teacher must use his own judgment about the amount of time to be spent on familiarizing the children with such terms as *subtrahend, minuend, dividend,* etc. These terms are certainly still in use, though not as a basic vocabulary.

A popular standardized achievement test asks: "In which of the following examples is 10 used as a product?

(a) $2 \times 5 = 10$
(b) $10 \div 2 = 5$"

If the child answers (b), his answer will be marked incorrect. Children ought to know of this situation and be advised of the consequences.

The words, "quotient as the missing term" in a division problem is used by some to help formulate division problem is used by some to help formulate statements such as: "What is the quotient in the following: $\frac{\quad}{3)42}$

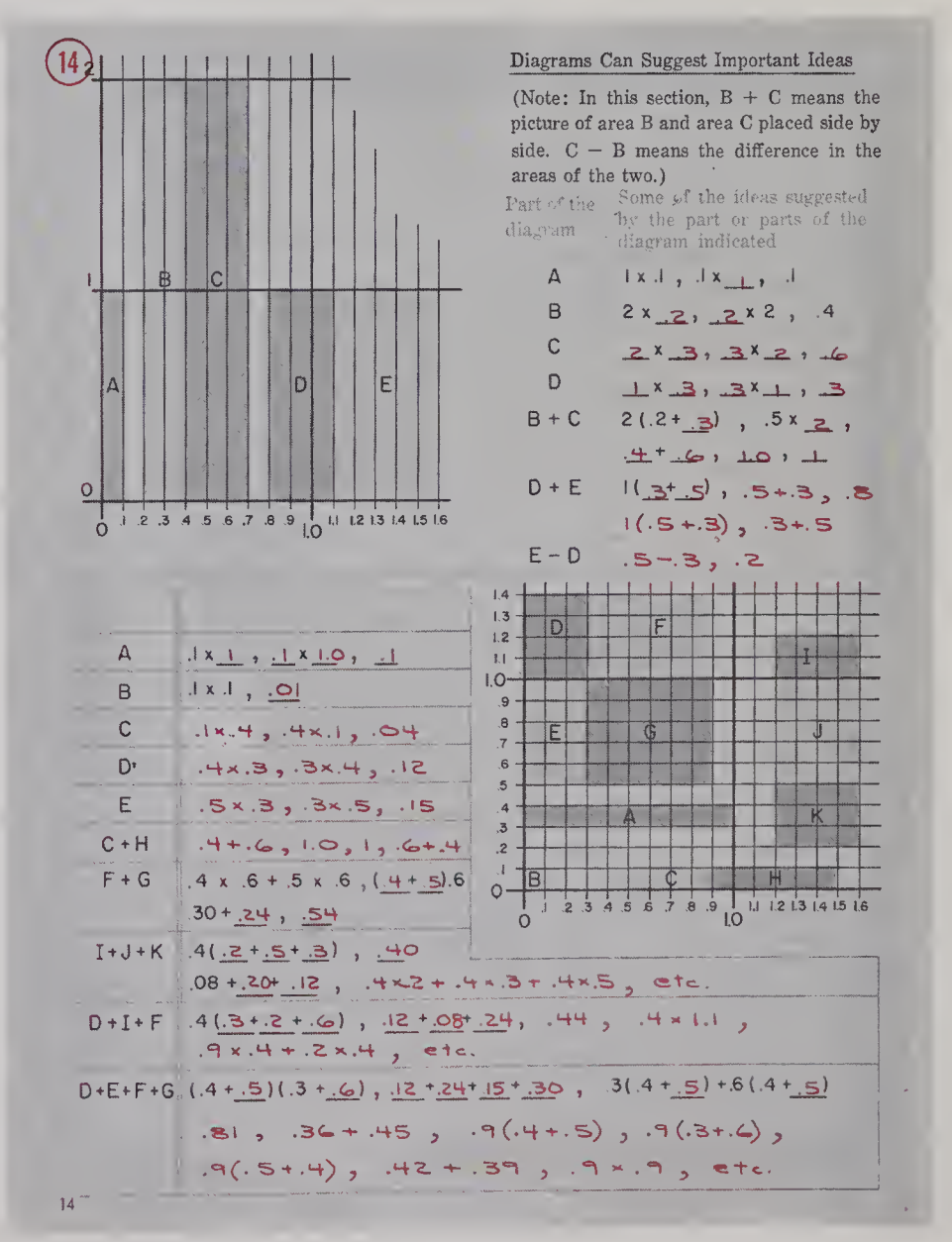
Also, "What is the divisor?" Seldom, "What is the dividend?" But these are a supplementary vocabulary and can be taught as such.

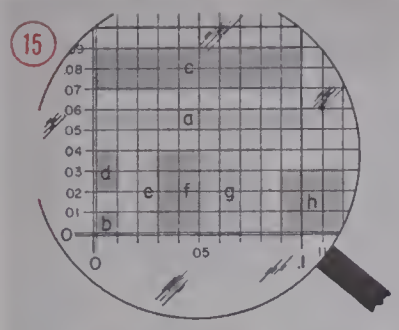
However, "addend," "sum," "factor" and "product" are words needed for the basic vocabulary of arithmetic.

Notes to accompany *Work Text* pages 14-15

Work Text pages 14 and 15 expand the old idea reintroduced on page 12. This is, of course, review for most children. However, we are not only concerned with whether or not they remember the ways we have developed to "manipulate the decimals." Too often, the rules are on recall, but not the ideas behind the rules.

Talking about the sketches on pages 14 and 15, and similar sketches on the board is a most important activity at this point. That $1/10 \times 1/10 = 1/100$ or that $.1 \times .1 = .01$ may be an instance of applied procedures that have lost their meaning. However, it is quite another situation to ask: "What part of the sketches suggests the idea that $.1 \times .1 = .01$? Is the idea pointed out in two of the sketches?" (*B* in the sketch on page 14 and the bulk of the sketch on page 15.)





Under a Magnifying Glass—

This sketch indicates how the little blue block B on the previous page might look under a strong magnifying glass. It is a sketch to suggest that .1 × .1 = .01.

On each number line shown, .1 is divided into 10 parts. Each then is $\frac{1}{10}$ of $\frac{1}{10}$ or $\frac{1}{100}$ or .01.

Complete this little Adding Machine

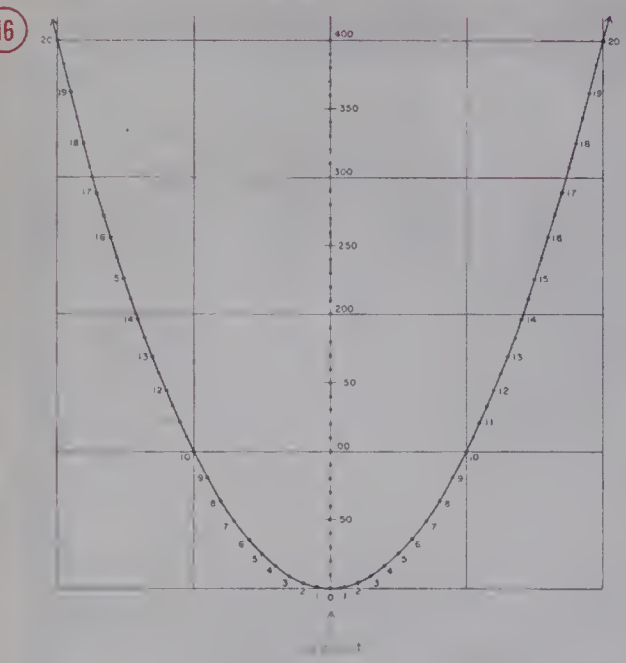
+	3	.4	2.5	.03
4	7	4.4	6.5	4.03
.3	3.3	.7	2.8	.33
.9	3.9	1.3	3.4	.93
3.7	6.7	4.1	6.2	3.73
.16	3.16	.56	2.66	.19
1.6	4.6	2.0	4.1	1.63
.54	3.54	.94	3.04	.57
2.5	5.5	2.9	5.00	2.53
.07	3.07	.47	2.57	.10
.67	3.67	1.07	3.17	.70
2.79	5.79	3.19	5.29	2.82
1.97	4.97	2.37	4.47	2.00

a.	.01 x .1 , .1 x .01 , .001
b	.01 x .01 , .0001
c	.02 x .1 , .1 x .02 , .002
d	.02 x .01 , .01 x .02 , .0002
e	.01 × .04 , .04 × .01 , .0004
f	.02 × .04 , .04 × .02 , .0008
g	.03 × .04 , .04 × .03 , .0012
h	.03 × .03 , .0009
b + h	.0001 + .0009 , .0010 , .001
b + d	.0001 + .0002 , .0003
a + c	.001 + .002 , .003 , .0030
a + e	.001 + .0004 , .0014
f + g	.04(.02 + .03) , .0020
	.0008 + .0012
e + f + g	(.01 + .02 + .03).04 , .0024
	.0004 + .0008 + .0012

A Multiplying Machine

×	4	.4	.04	1.5	3.25	2.01	.35
2	8	.8	.08	3.0	6.50	4.02	.70
.6	2.4	.24	.024	.90	1.950	1.206	.210
.2	.8	.08	.008	.30	.650	.402	.070
.01	.04	.004	.0004	.015	.0325	.0201	.0035
.12	.48	.048	.0048	.180	.3900	.2412	.0420

15



When two factors are given, find one on each of the lines that curve up and out from 0. Connect these points with a straight-edge. At what point does it seem to cut the center scale?

Where one factor is missing, lay a straight-edge across the graph through the point indicating the given factor and the product. Notice where the straight-edge cuts the other curved line.

If you work carefully, you can estimate the size of the missing factor or product within a small amount.

In the statement, 12 × 15 = 180, we call 12 and 15 the "factors," and we call 180 the "product" of those factors. Is it true that, if any two of the three elements of such a statement are given, we can find the third? Yes

one factor	15	9	12	17	18
another factor	10	12	16	17	15
product	150	108	192	289	270

The incompleted statement, 162 ÷ 18 ___ indicates that the product and one of the factors is given. To complete it, the missing factor must be found.

Find the missing factor in each:

12 x 12 = 144 306 ÷ 18 = 17

13 = 182 ÷ 14 13 x 8 = 104

	using the graph	by computation
12 x 16 = ___		192
___ x 17 = 187		11
13 x ___ = 195		15
247 ÷ 19 = ___		13
324 ÷ ___ = 18		18
___ ÷ 19 = 12		228
10.5 x 18 = ___		189
12 x 17.5 = ___		210
216 ÷ 16 = ___		13.5
___ ÷ 8 = 19.5		156
10.5 x 10.5 = ___		110.25
361 ÷ 19 = ___		19
13 x 130 = ___		1,690
1.5 x 17 = ___		25.5
1.3 x 1.6 = ___		2.08
297 ÷ 16.5 = ___		18
158 ÷ ___ = $12\frac{2}{13}$		13

"What idea is suggested by the green area labeled *K* on page 14? . . . by the blue area labeled *b* on page 14?"

"In what way does the blue area labeled *F* suggest the idea of .6 × .4?"

The sketches are not given as a crutch or an aid in finding answers. They are included as a means of directing attention to a meaning of each example—a representation suggesting the idea of decimal fractions and the idea of multiplication of decimal fractions.

Notes to accompany *Work Text* pages 16-17

In addition to reviewing the idea of multiplication and its inverse, division, *Work Text* pages 16 and 17 involve the use of "scales."

Each diagram is drawn with considerable accuracy, so estimates can be made very accurately, and a little ingenuity can extend this accuracy. A ruler or straight-edge of any kind (such as the edge of a sheet of paper) is all that's needed to convert these diagrams into useful "multiplying and dividing machines." (The technical name for this kind of devise is *nomograph.)*

To find the product of 12 and 15, locate one factor on one curved scale and the other factor on the other. Connect these points with a straight-edge. At what point do they cross the vertical Scale? . . . at 180. By computation, we find that 12 × 15 = 180.

When the factors are 12 and 16, the straight line crosses the scale a little above 190. A quick glance at the two factors tells us that the left-hand digit of the product must be 2. So, we must decide whether the product is 182, 192 or 202—since it is close to 190. We can see clearly that the point is a little above 190 and certainly less than 200. So, we conclude that 12 × 16 = 192 (and computation bears us out).

One example asks for the product of 13 and 130. But each curved scale goes no further than 20. So, we restate the question as: 13 × 13 × 10 = ____. We find that line connecting 13 on one curved scale and 13 on the other, crosses the center scale very close to 170. The product of 13 and 13 must end in 9, and we conclude that 13 × 13 = 169 and 169 × 10 = 1690.

Next, consider the points on the curved scales marked between the points labeled with whole number symbols (numerals). How much is 10 1/2 ×

11 1/2? We can see that the connecting line would pass very close to 120. Calculation tells us that 10 1/2 × 11 1/2 = 120 1/4—very close indeed. But, we have now encountered one of the weaknesses of the devise. Under such circumstances, there is very little information readily available to tell us more than what we see—that the cross point is very close to 120 on the scale. If we require greater accuracy, then we must compute. However, if we are thinking about the area of a room 10 1/2′ by 11 1/2′, then the conclusion that there are about 120 square feet of area would be close enough for most purposes. When human beings make such scales, they are less than perfect scales. When human beings read those imperfect scales, their readings are imperfect. So "close" is an idea we must deal with, and we must develop a feeling about what does "close enough" mean in terms of the particular situation at hand. (This is, of course, exactly the problem engineers and mathematicians face when they use a slide rule which, at best, gives results that are "close.")

Another example asks for the product of 1.3 and 1.6, and we find we are down in a section of our machine that is quite hard to read. What should we do? One approach is to restate the problem this way:

$$13/10 \times 16/10 = \frac{13\times16}{100}, \text{ or}$$

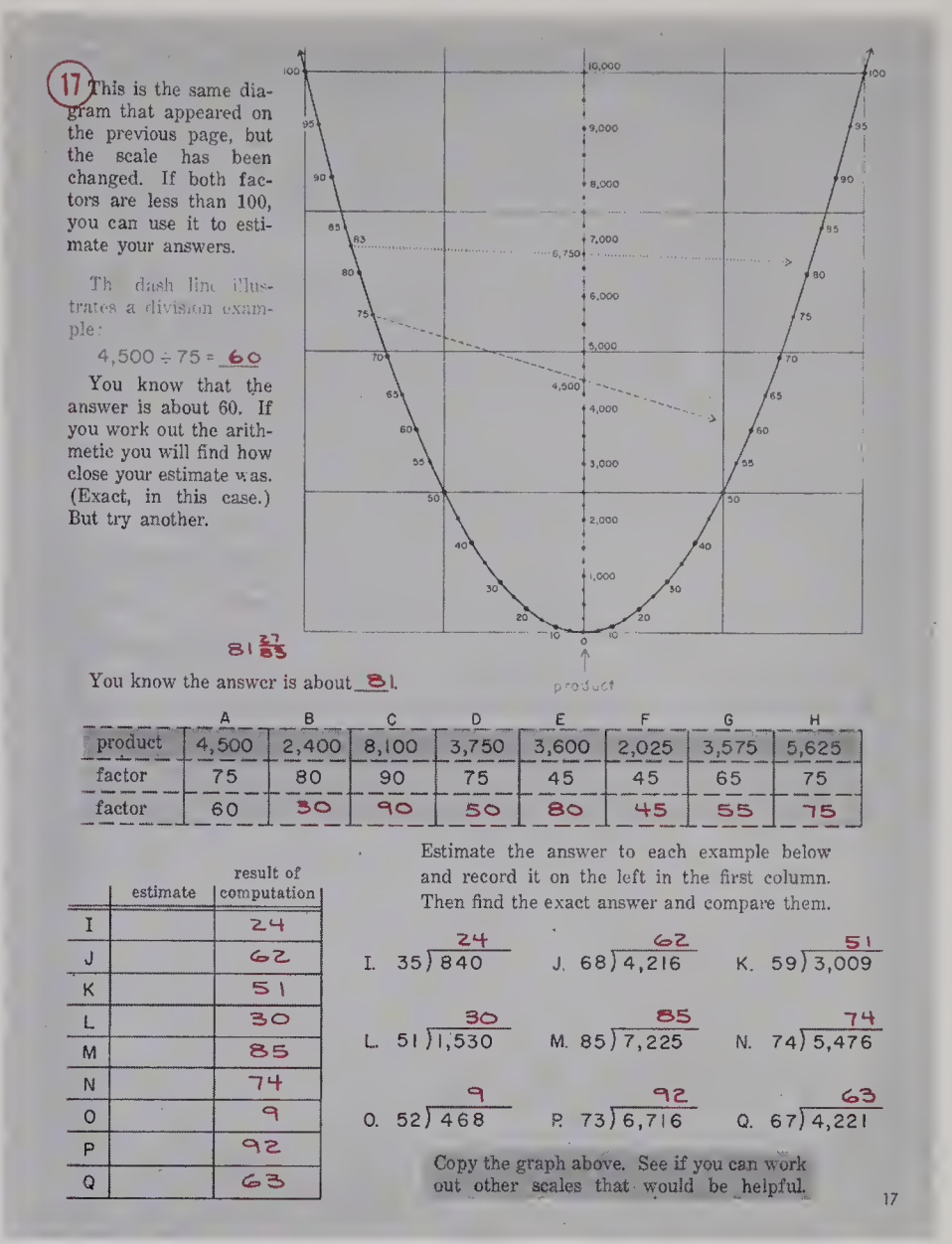
$$1.3 \times 1.6 = (13\times16) \div 100$$

We can find that 13 × 16 is a little less than 210, and since the last digit must be 8, we conclude it is 208. We know that 208 ÷ 100 = 2.08, so we answer that 1.3 × 1.6 = 2.08. Why do we not say "close to 2.08?" Because, along the way, we eliminated the "close" limitation by computing the "last digit."

A very similar diagram appears on page *Work Text* 17, only the scales have been changed and "close" has a new value scale.

A most important aspect of learning to use these diagrams is that they are as useful in finding missing factors as in finding missing products. "How much is 108 ÷ 9?" is one of the early questions posed. Find 9 on one of the curved scales; find 108 on the vertical scale; connect the points and extend the line. This line crosses the other curved scale at "about 12." Computation would show that 108 ÷ 9 = 12, or 9 × 12 = 108.

The fact that this diagram can be used both as a "multiplying machine" and a "dividing machine" is a quite dramatic demonstration that these two operations have a lot in common. This would be less convincing if we sometimes looked for an estimate of a "product" on the vertical line and an estimate of a "dividend" in another example. Further, we would divert attention from the important question if—in searching for the product of 12 and 16—we became involved in trying to decide whether to locate 12 or 16 on the curved scale on the left side of the diagram, when the most important idea is that it doesn't make any difference, as long as you locate 12 and 16 on different curved scales.



17 This is the same diagram that appeared on the previous page, but the scale has been changed. If both factors are less than 100, you can use it to estimate your answers.

Th dash line illustrates a division example:

4,500 ÷ 75 = 60

You know that the answer is about 60. If you work out the arithmetic you will find how close your estimate was. (Exact, in this case.) But try another.

81 27/85

You know the answer is about 81.

	A	B	C	D	E	F	G	H
product	4,500	2,400	8,100	3,750	3,600	2,025	3,575	5,625
factor	75	80	90	75	45	45	65	75
factor	60	30	90	50	80	45	55	75

	estimate	result of computation
I		24
J		62
K		51
L		30
M		85
N		74
O		9
P		92
Q		63

Estimate the answer to each example below and record it on the left in the first column. Then find the exact answer and compare them.

I. 24 / 35)840 J. 62 / 68)4,216 K. 51 / 59)3,009

L. 30 / 51)1,530 M. 85 / 85)7,225 N. 74 / 74)5,476

O. 9 / 52)468 P. 92 / 73)6,716 Q. 63 / 67)4,221

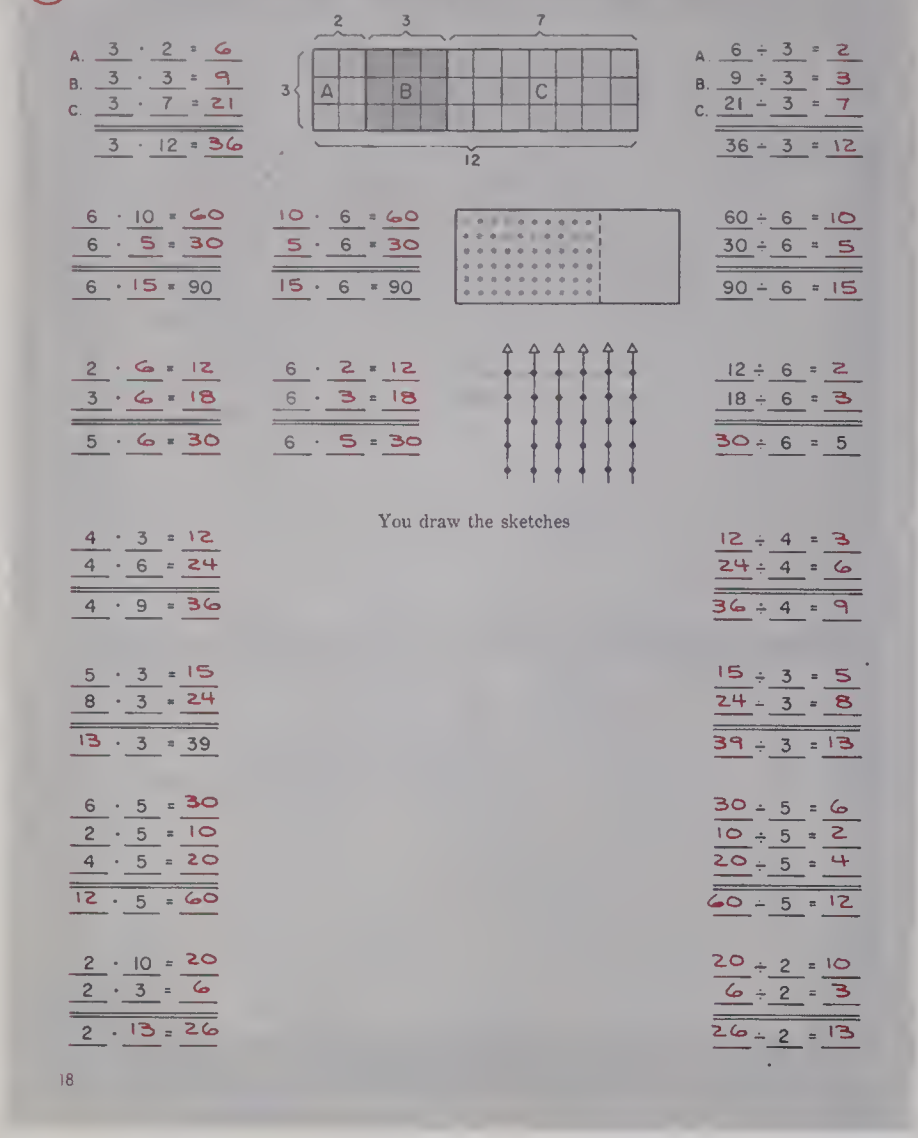
Copy the graph above. See if you can work out other scales that would be helpful.

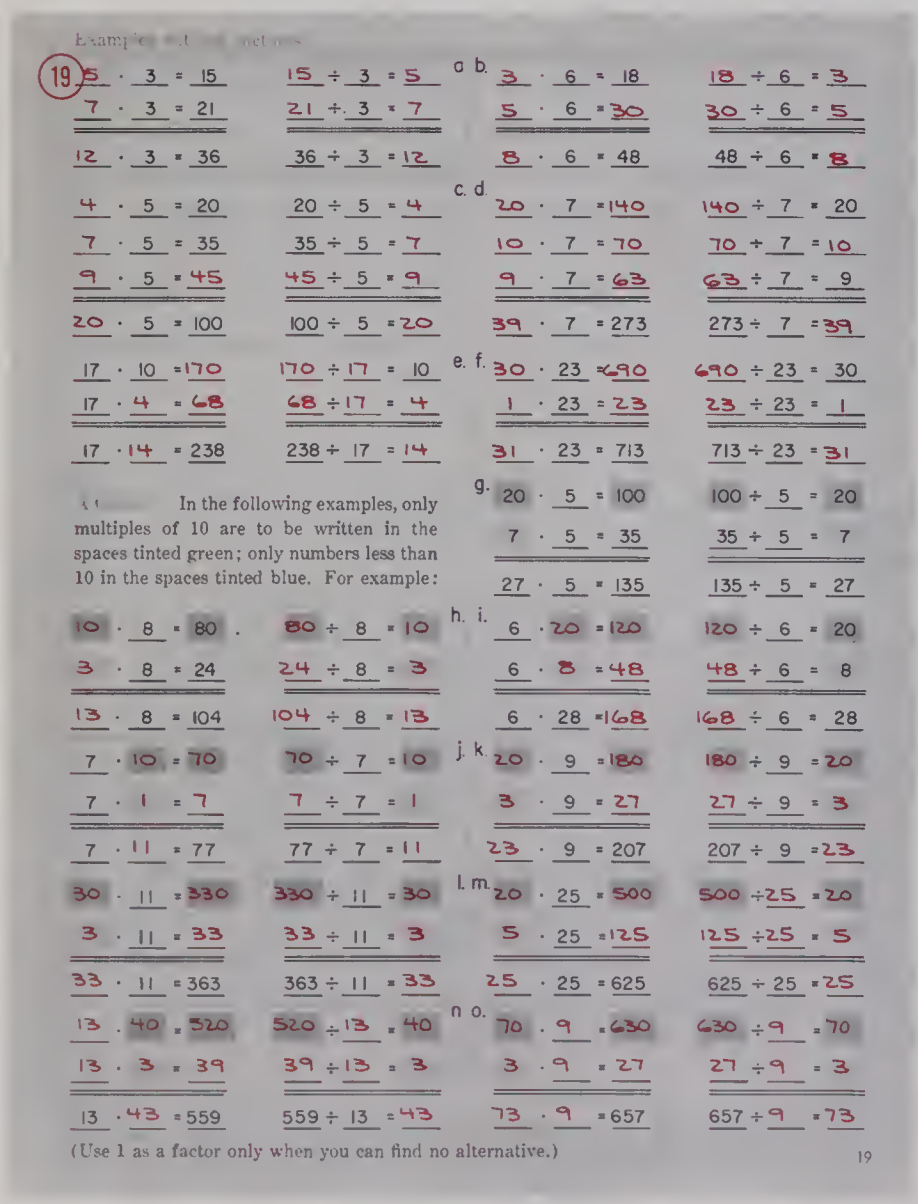
17

Notes to accompany *Work Text* pages 18-20

Sketches again introduce an idea—or reintroduce it. In the development on *Work Text* pages 18, 19 and 20, we are considering the principle of regrouping that underlies the algorisms of multiplication and division.

Pages 18 and half of 19 depend on examples in which the regrouping is completely general. The other half of page 19 and all of 20, through a game-like device, focus on a regrouping that is particularly convenient in our decimal system of numeration. However, the basic principle is the same in both.





While addition and multiplication live separate lives part of the time, they are married in this way:

$$(3\times8) + (3\times2) = 3 \times (8+2)$$
$$24 \quad + \quad 6 \quad = 3 \times \ 10$$

An example of the same idea is:

$$5 \times (7+8) = (5\times7) + (5\times8)$$
$$5 \times \ 15 \quad = \ 35 \quad + \ 40$$

An expressive title for this pattern might be:

$$a \times (b+c) = (a\times b) + (a\times c)$$

Mathematicians have named this pattern the "distributive law" of multiplication over addition. We use it as a basic device in our algorism of multiplication (and division):

$$7 \times 19 = 7 \times (9+10) = 7 \times 9 + 7 \times 10$$
$$= 63 \quad + 70 \quad = 133$$

Written in a different form:

$$\begin{array}{r} 19 \\ \times\ 7 \\ \hline 133 \end{array} \quad \longrightarrow \quad \begin{array}{r} 10 + 9 \\ \times\ 7 \\ \hline 70 + 63 \end{array}$$

We regroup the 19 into 10 + 9 or 9 + 10 – and then apply the distributive law. In *Level D Work Text,* a character, Ben Harris, used this nice pattern and called it his "Do-It-to-Both" rule because he saw it this way:

$$3 \times (4+5) = 3\times4 + 3\times5$$
$$= \ 12 \ + \ 15$$
$$= \ 27$$

Work Text page 18 starts off with a sketch suggesting an expansion of this idea:

$$3 \times 12 = 3 \times (2+3+7) = 3\times2 + 3\times3 + 3\times7$$
$$= \ 6 \ + \ 9 \ + \ 21$$
$$= \ 36$$

(Ben called this his "Do-It-to-All-Three" version.)

And, there is a similar pattern we use in division that is simply a reverse statement of the idea. It is suggest by the same sketch:

$$36 \div 3 = (6+9+21) \div 3 = 6 \div 3 + 9 \div 3 + 21 \div 3$$
$$= 2 \quad + 3 \quad + \quad 7$$
$$= 12$$

Written in another form, this might look like:

$$\overline{3)36} = \overline{3)6+9+21}\ \begin{matrix} 2+3+7 = 12 \\ \text{and } 2+3+7 = 12 \end{matrix} \quad \text{or } \overline{3)6} + \overline{3)9} + \overline{3)21}$$

We use this idea of regrouping in division, but it often gets lost in the mechanics. What we really do is indicated by the following:

$$\overline{3)228} = \overline{3)210+18}\ 70+6 = 76$$

Of course, we might have done the regrouping in other ways:

$$\overline{3)228} = \overline{3)105+75+48}\ 35+25+16 \quad \text{or} \quad \overline{3)240-12}\ 80-4 = 76$$

But we select a certain kind of regrouping because it is most convenient in the use of our "base 10" or decimal system of numeration.

As soon as we impose the limitations of the division game on page 19, all results must be written in a very thinly disguised form of the traditional algorism.

This approach to regrouping moves right on to include situations in which one of the factors is not a whole number – when remainders are involved. In a somewhat different form than shown, the first example on page 20 would look like this:

$$\overline{8)148} = \overline{8)80+64+4}\ 10+8+4/8 = 18\ 4/8 = 18\ 1/2$$

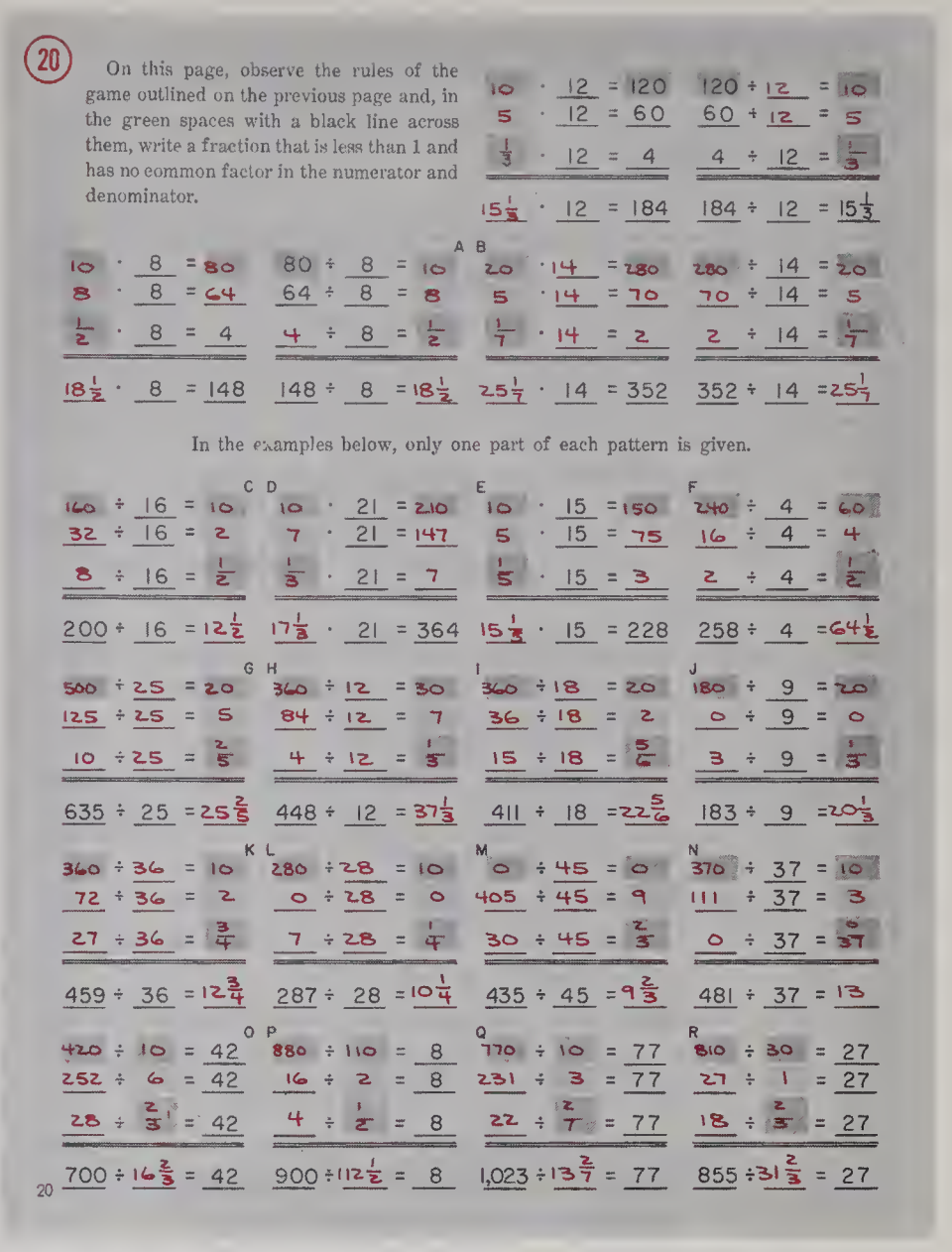
One way to read 4/8 is "4 divided by 8," so $8 \div 4 = 4/8 = 1/2$. Thus, "remainders" are a logical extension of the way we analyze a division problem.

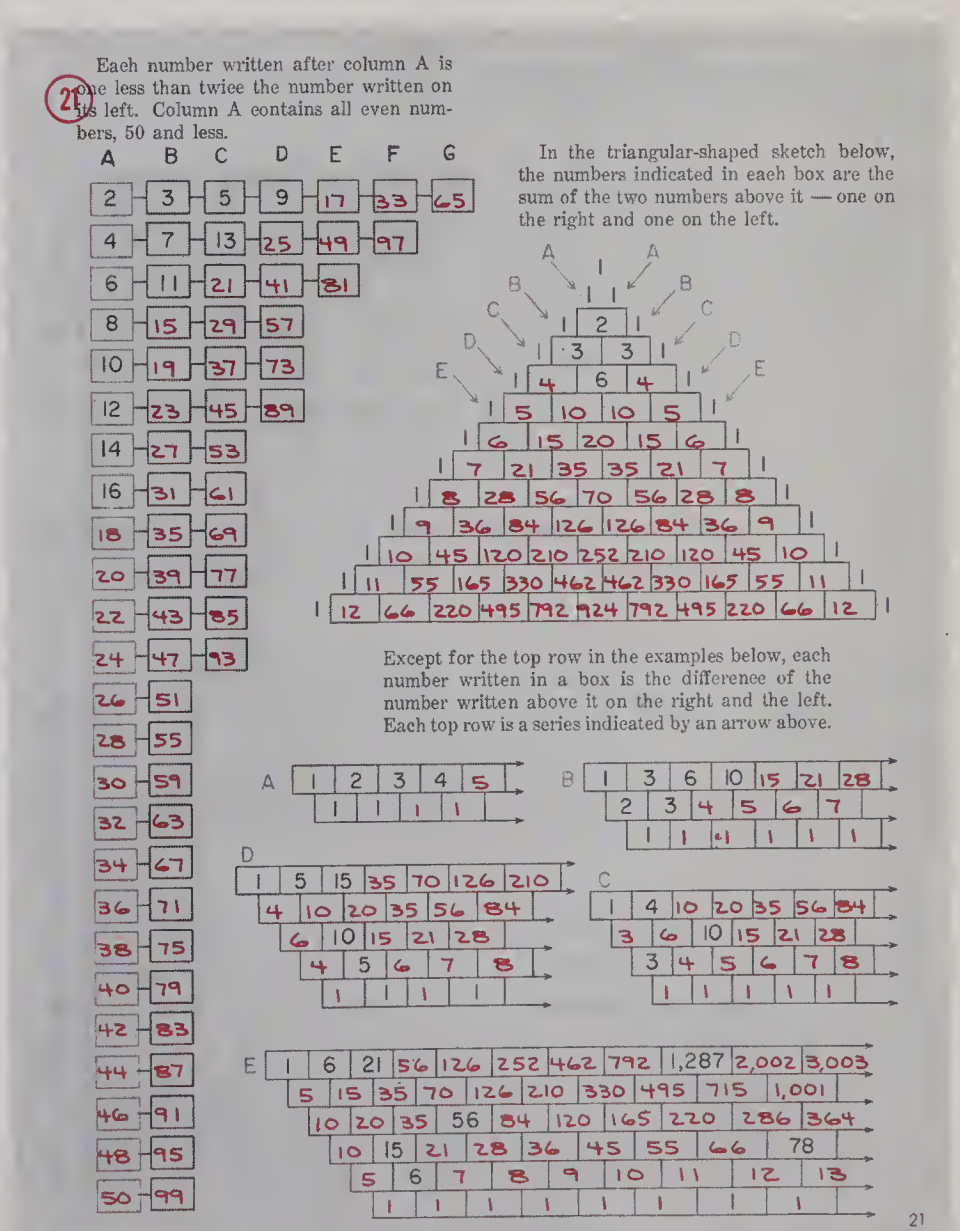
Notes to accompany *Work Text* page 21

Two unusual patterns are explored on page 21. On the left, a very simple rule is given in two parts:

a. In the left hand column of cells, each number noted is the next even number (moving down the page);

b. In all other cells, the number noted is "one less than twice the number noted in the cell on its left."


20
On this page, observe the rules of the game outlined on the previous page and, in the green spaces with a black line across them, write a fraction that is less than 1 and has no common factor in the numerator and denominator.
In the examples below, only one part of each pattern is given.
A B C D E F G H I J K L M N O P Q R
20


21
Each number written after column A is one less than twice the number written on its left. Column A contains all even numbers, 50 and less.
A B C D E F G
In the triangular-shaped sketch below, the numbers indicated in each box are the sum of the two numbers above it — one on the right and one on the left.
Except for the top row in the examples below, each number written in a box is the difference of the number written above it on the right and the left. Each top row is a series indicated by an arrow above.
A B C D E
21

When the cells are filled in, there are some interesting facts to note:

1. No number has been repeated; no number less than 100 is omitted. If the pattern were extended according to the rules above, would numbers ever be repeated? If so, what is the first repetition? (There never would be a repetition.)

2. Begin with G and study each column:
   G—Contains all numbers less than 100 that leave a remainder of 1 when divided by *64*.
   F—Contains all numbers less than 100 and not listed in G that leave a remainder of 1 when divided by *32*.
   E—Contains all numbers less than 100 not listed in F or G that leave a remainder of 1 when divided by *16*.
   D—Contains all numbers less than 100 not listed in E, F or G that leave a remainder of 1 when divided by *8*.
   C—Contains all numbers less than 100 not listed in D, E, F and G that leave a remainder of 1 when divided by *4*.
   B—Contains all numbers less than 100 not listed in C, D, E, F or G that leave a remainder of 1 when divided by *2*.

That pattern is a familiar pattern, but it ought to be extended once more:

64 32 16 8 4 2 ?

The logic of this development would lead us to say:

A—Contains all numbers less than 100 not listed in B, C, D, E, F or G that leave a remainder of 1 when divided by 1 (from the pattern above).

Does this make sense? Does $2 \div 1$ leave a remainder of 1? We could write those divisions this way:

$$1\overline{)2}\quad 1 \qquad 1\overline{)4}\quad 3 \qquad 1\overline{)6}\quad 5 \qquad 1\overline{)8}\quad 7 \qquad 1\overline{)10}\quad 9$$
$$1 \qquad\quad 1 \qquad\quad 1 \qquad\quad 1 \qquad\quad 1$$

Another simple rule leads to the construction of an interesting triangle. Each cell contains a sum of the numbers written above it, right and left. Looking down the diagonals indicated by arrows, we notice that each is larger than the one before it. Each series is analyzed in a table below. We are concerned with the way in which each series grows. So, we note below each successive pair the amount of growth. In *B*, we note that 1 and 3 represent a growth of 2; 3 and 6 represent a growth of 3; etc.

In each case, the row below each chart would show a string of zeroes: $1 - 1 = 0$, $1 - 1 = 0$, etc. We notice that each succeeding study requires another line of "differences," before the growth is constant. A has 2 lines; B, 3 lines; C, 4 lines; D, 5 lines; and E, 6 lines.

Some pupils may want to make the triangle grow bigger and bigger by adding another row on the base. And then study the diagonals that would be labeled F, G, H, etc.

This activity is a gold mine of patterns. It will be considered many times in high school and college and graduate school. The patterns were studied intensively by Pascal, and from that study we have "Pascal's Triangle."

Notes to accompany *Work Text* pages 22-26

Each of the next group of pages—22 through 26—leads to a "pattern hunt." Part of the activity in each hunt is the search for an appropriate name or title that seems to express the idea of the pattern, or point it out.

Examples on these five pages are all of a very similar kind. They depend on the form

____, ____ → ____

The idea is: given a pair of numbers (____,____) and a "secret rule," when the rule is applied to the pair it points (→) to a particular third number (____). For example:

3, 2 → 6

We can read this, "When a secret rule is applied to 3 and 2, it leads to 6"—or, in short, "three and two goes to 6."

You suspect the rule is simply "multiply the two numbers to find the product—the third number." But you can't be sure. Some more examples show your uncertainty was justified:

3, 2 → 6
3, 3 → 7
2, 7 → 10

Your first inclination was to see the pattern as:

$$a, b \rightarrow a \times b$$

But it turns out to be:

$$a, b \rightarrow a + b + 1$$

These indications of patterns look quite like the algebra most everyone has had in high school And some children will ask: "Is this algebra?" You dare not say "No," and no good cause would be served by a "Yes."

Instead, beg the question with, "Whatever it's called, we are simply pointing to patterns we see."

Perhaps children would like to experiment with other ways of pointing out the patterns:

Add the numbers noted on the left of the arrow and add one to that sum. That final result is written on the right of the arrow.

. . .but that's a very long name. Have the class try to condense the name:

Add the numbers on the left and add 1 to find the number on the right.

Anyway, it's shorter, You might try shorthand of a kind:

Add Left Nos. + 1. This goes to right No.

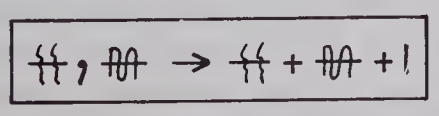
Still shorter. Keep working at it, and you will come to the conclusion that:

$$x, y \rightarrow x + y + 1$$

is a nice compact way to point to the patterns. If the letters seem mysterious, try "frames:"

$$\square, \triangle \rightarrow \square + \triangle + 1$$

Or, invent some new symbols:



22

Here is one of the many pictures that suggests the idea of addition and subtraction.

This picture suggests the idea just as well as those below. Each expression is simply another way of saying the same thing: we are arranging the addends and sum in a different order.

$7 + 2 = 9$ $\quad 9 - 7 = 2$

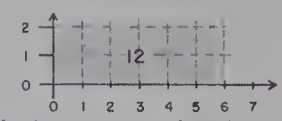
$2 + 7 = 9$ $\quad 9 - 2 = 7$

And, by switching the "addition" and "equal" signs, we change the form but not the meaning:

$9 = 7 + 2$ $\quad 2 = 9 - 7$

$9 = 2 + 7$ $\quad 7 = 9 - 2$

Here is one of the many pictures that suggests the idea of multiplication and division.



The factors are 2 and 6; the product is 12. (If you are in doubt, count the small blocks in the blue-tinted area.) This picture illustrates all of the following:

$2 \times 6 = 12$ $\quad 12 \div 6 = 2$

$6 \times 2 = 12$ $\quad 12 \div 2 = 6$

And, by switching the "multiplication" and "equal" signs, we change the form but not the meaning:

$12 = 6 \times 2$ $\quad 2 = 12 \div 6$

$12 = 2 \times 6$ $\quad 6 = 12 \div 2$

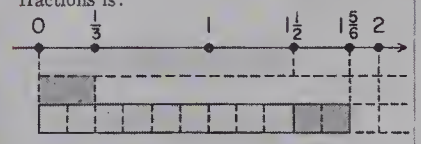
Pattern Hunt

7, 0 → 7	2, 2 → 4	4, 2 → 2	8, 0 → 8
19, 91 → 110	1, 1 → 1	9, 3 → 3	4, 2 → 2
6, 8 → 14	0, 0 → 0	70, 10 → 7	19, 4 → 15
65, 65 → 130	8, 3 → 24	42, 6 → 7	74, 47 → 27
42, 72 → 114	9, 12 → 108	96, 12 → 8	92, 38 → 54
$a, b \rightarrow a + b$	$m, n \rightarrow m \cdot n$	$x, y \rightarrow x \div y$	$r, s \rightarrow r - s$
$\frac{1}{2}, \frac{1}{4} \rightarrow \frac{3}{4}$	→	→	→
.2, .3 → .5	→	→	→
sum	product	missing factor	missing addend
5, 11 → 6	3, 12 → 4	17, 3 → 51	2, 4 → 2
9, 21 → 12	5, 45 → 9	5, 11 → 55	0, 0 → 0
31, 50 → 19	1, 71 → 71	12, 12 → 144	1, 1 → 0
0, 19 → 19	14, 42 → 3	10, 10 → 100	13, 13 → 0
$a, x \rightarrow x - a$	$m, r \rightarrow r \div m$	$d, t \rightarrow d \cdot t$	$x, y \rightarrow y - x$
→	→	20, 20 → 400	17, 63 → 46
→	→	30, 40 → 1,200	$\frac{1}{3}, 1 \rightarrow \frac{2}{3}$
addend	factor	missing product	missing addend

22

23

A picture that suggests an example of the idea of addition and subtraction of fractions is:

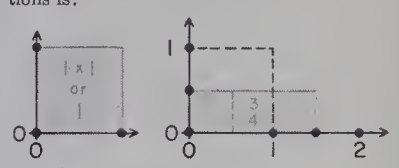


This picture suggests an idea that can be expressed in many ways. The form of the sentence changes, but not its meaning:

$1\frac{1}{2} + \frac{1}{3} = 1\frac{5}{6}$ $\quad 1\frac{5}{6} - \frac{1}{3} = 1\frac{1}{2}$

$\frac{1}{3} + 1\frac{1}{2} = 1\frac{5}{6}$ $\quad 1\frac{5}{6} - 1\frac{1}{2} = \frac{1}{3}$

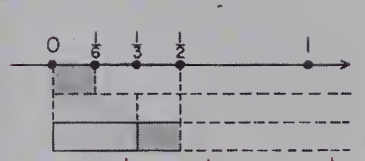
A picture that suggests an example of the idea of multiplication and division of fractions is:



If the sketch on the left above suggests that $1 \times 1 = 1$, then the other sketch suggests that $\frac{1}{2} \times 1\frac{1}{2} = \frac{3}{4}$. Changing the sentence form involving these factors and products, we say:

$\frac{1}{2} \times 1\frac{1}{2} = \frac{3}{4}$ $\quad \frac{3}{4} \div \frac{1}{2} = 1\frac{1}{2}$

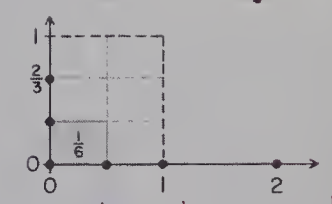
$1\frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$ $\quad \frac{3}{4} \div 1\frac{1}{2} = \frac{1}{2}$



Addends are $\frac{1}{6}$ and $\frac{1}{3}$. Sum is $\frac{1}{2}$. Various ways of expressing the idea suggested are:

$\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$ $\quad \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$ $\quad \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$



Factors are $\frac{1}{3}$ and $\frac{1}{2}$. Product is $\frac{1}{6}$.

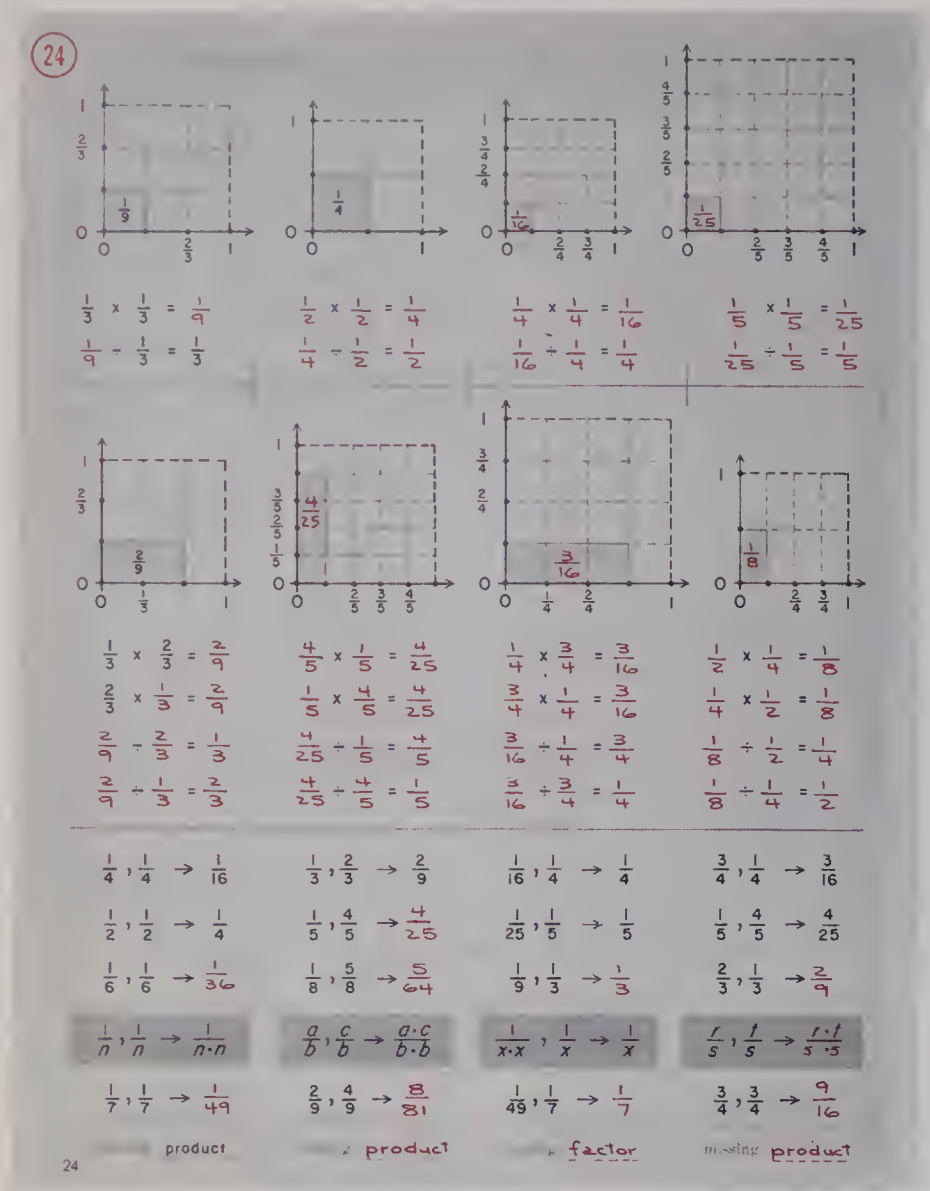
$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$ $\quad \frac{1}{6} \div \frac{1}{2} = \frac{1}{3}$

$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ $\quad \frac{1}{6} \div \frac{1}{3} = \frac{1}{2}$

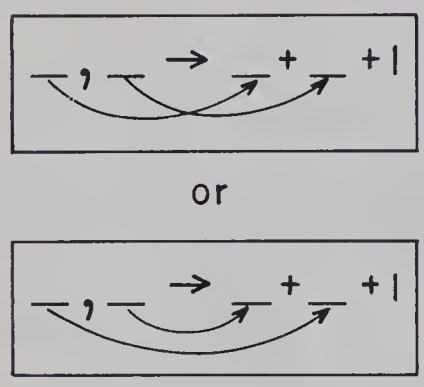
Pattern Hunt

$2\frac{1}{2}, 1\frac{1}{2} \rightarrow 4$	3, 2 → $1\frac{1}{2}$	$\frac{1}{2}, \frac{1}{3} \rightarrow \frac{1}{6}$	6, 8 → 2
3, 8 → 11	75, 15 → 5	0, $\frac{7}{8} \rightarrow 0$	$\frac{1}{3}, 1 \rightarrow \frac{2}{3}$
19, 17 → 36	144, 12 → 12	7, 19 → 133	$1\frac{1}{2}, 8 \rightarrow 6\frac{1}{2}$
$1\frac{1}{2}, \frac{1}{3} \rightarrow 1\frac{5}{6}$	$\frac{3}{4}, \frac{1}{2} \rightarrow 1\frac{1}{2}$	$1\frac{1}{2}, \frac{1}{2} \rightarrow \frac{3}{4}$	$\frac{1}{3}, 1\frac{5}{6} \rightarrow 1\frac{1}{2}$
58, 94 → 152	$\frac{1}{6}, \frac{1}{3} \rightarrow \frac{1}{2}$	15, 15 → 225	$\frac{1}{3}, \frac{1}{2} \rightarrow \frac{1}{6}$
$a, n \rightarrow a + n$	$x, y \rightarrow x \div y$	$v, t \rightarrow v \cdot t$	$c, d \rightarrow d - c$
$\frac{1}{3}, 1\frac{1}{2} \rightarrow 1\frac{5}{6}$	$\frac{3}{4}, 1\frac{1}{2} \rightarrow \frac{1}{2}$	$\frac{1}{3}, \frac{1}{2} \rightarrow \frac{1}{6}$	255,500 → 245
$\frac{1}{3}, \frac{1}{6} \rightarrow \frac{1}{2}$	132, 11 → 12	13, 13 → 169	.9, 1 → .1

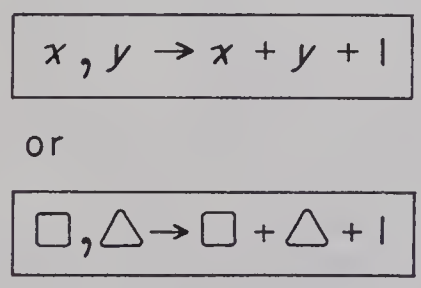
23



And here are other ways to note the idea:



Keep trying. Each new effort helps expalin the pattern. And any one we select is quite adequate. It's hard to improve on



and that's all there is to it. We accept or reject a title or name for the pattern solely on our judgment of whether or not it points to the idea—and we prefer brief names.

Now, take another step to deepen this idea of a pattern and a name for it. Consider the same examples:

$3, 2 \rightarrow 6$
$3, 3 \rightarrow 7$
$2, 7 \rightarrow 10$
$8, 0 \rightarrow 9$
$0, 0 \rightarrow 1$
$1, 1 \rightarrow 3$

$r, s \rightarrow r + s + 1$

Suppose we think of that name as a headline. What are some stories that could appropriately accompany it?

"After I spent 3¢ for a pencil and 2¢ for a pencil, I had 1¢ left. How much did I start with?"

"Three rocket motors were used for take-off and 2 more to achieve orbit speed. One motor was held in reserve for guidance and for reverse thrust on reentry."

"Bill earned $2 last week and $7 this week. He now has $10 saved toward a chemistry set. He must have had only $1 saved before last week—or he spent some of the $9 he earned in the last two weeks."

Some may prefer stories almost as general as the headlines.

"I took some books back to the library and left some at home. The librarian said I was charged with one more than I can account for."

"Mary's age is one year more than the sum of the ages of her two brothers."

This kind of supplemental classroom development will contribute greatly to increased understanding of the idea of pattern indicators, relating each pattern to life experiences (real and imagined).

Simple facts expanded

Pages 22 and 23 start with a sketch that suggests the idea of addition and subtraction, and another sketch that suggests the idea of multiplication and division. And each of the patterns is developed by a group of simple facts, based on those operations. Pages 24, 25, and 26 expand these ideas beyond whole numbers to include fractions—including examples of division of one fraction by another. The "pattern indicators" begin to grow up.

As the first example is encountered on page 24,

$$\frac{1}{n}, \frac{1}{n} \rightarrow \frac{1}{n \cdot n}$$

suggest that children try to explain this idea any other way they like. Help them by explaining that any fraction with 1 as the numerator is called a "unit fraction." They may try a title such as:

> *The products of two like unit fractions is a unit fraction whose denominator is the denominator of the first mentioned fraction times itself.*

. . . which becomes awfully involved. One is not sure that the idea has been clearly pointed out, but don't labor the point—because our interest is in any name that seems to fit the pattern we want to point out.

(You may need to repeat that n · n = n × n and that this idea is sometimes written as n^2. They may also need help in reading the name given: "one over n and one over n goes to one over n times n.")

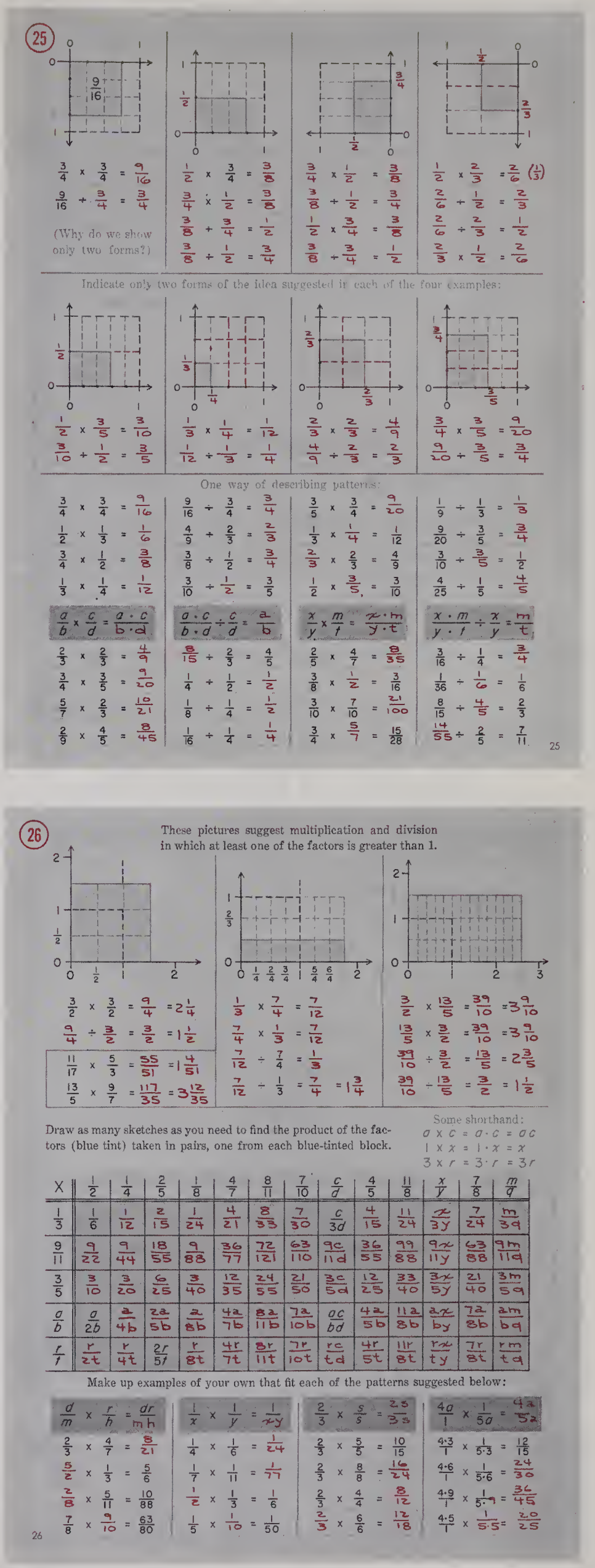
On page 25, the basic ____, ____ → ____ is changed, and the pattern hunt changes to a search for examples that fit a given pattern. Notice the similarity between the first two pattern indicators (left bottom) of page 25:

$$\underset{\text{factor}}{\frac{a}{b}} \times \underset{\text{factor}}{\frac{c}{d}} = \underset{\text{product}}{\frac{a \cdot c}{b \cdot d}}$$

$$\underset{\text{product}}{\frac{a \cdot c}{b \cdot d}} \div \underset{\text{factor}}{\frac{c}{d}} = \underset{\text{missing factor}}{\quad}$$

The first can be read: "a over b times c over d equals a times c over b times d." The second: "a times c over b times d divided by c over d equals a over b."

Most of the examples of the pattern involved are taken from those that accompany the sketches above. Any way that pupils use to complete the statements should not be questioned. They can draw out sketches for themselves—but very soon they will begin to rely on a pattern they see: that in each instance of multiplying one fraction by another, the numerator of the product is the product of the numerators of the factors: and that the same pattern applies to denominators. Which is simply a crude way of saying:


25
(Why do we show only two forms?)
Indicate only two forms of the idea suggested in each of the four examples:
One way of describing patterns:
26
These pictures suggest multiplication and division in which at least one of the factors is greater than 1.
Draw as many sketches as you need to find the product of the factors (blue tint) taken in pairs, one from each blue-tinted block.
Some shorthand:
Make up examples of your own that fit each of the patterns suggested below:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

Do not be concerned if many in the class cannot see all the way through to the last example on *Work Text* page 26. Pages ahead keep up the search for patterns and for concise ways to indicate them. Mastery of the process is a long way off. As it begins to mature, each child can go back, better equipped to fill out the series of examples he postponed.

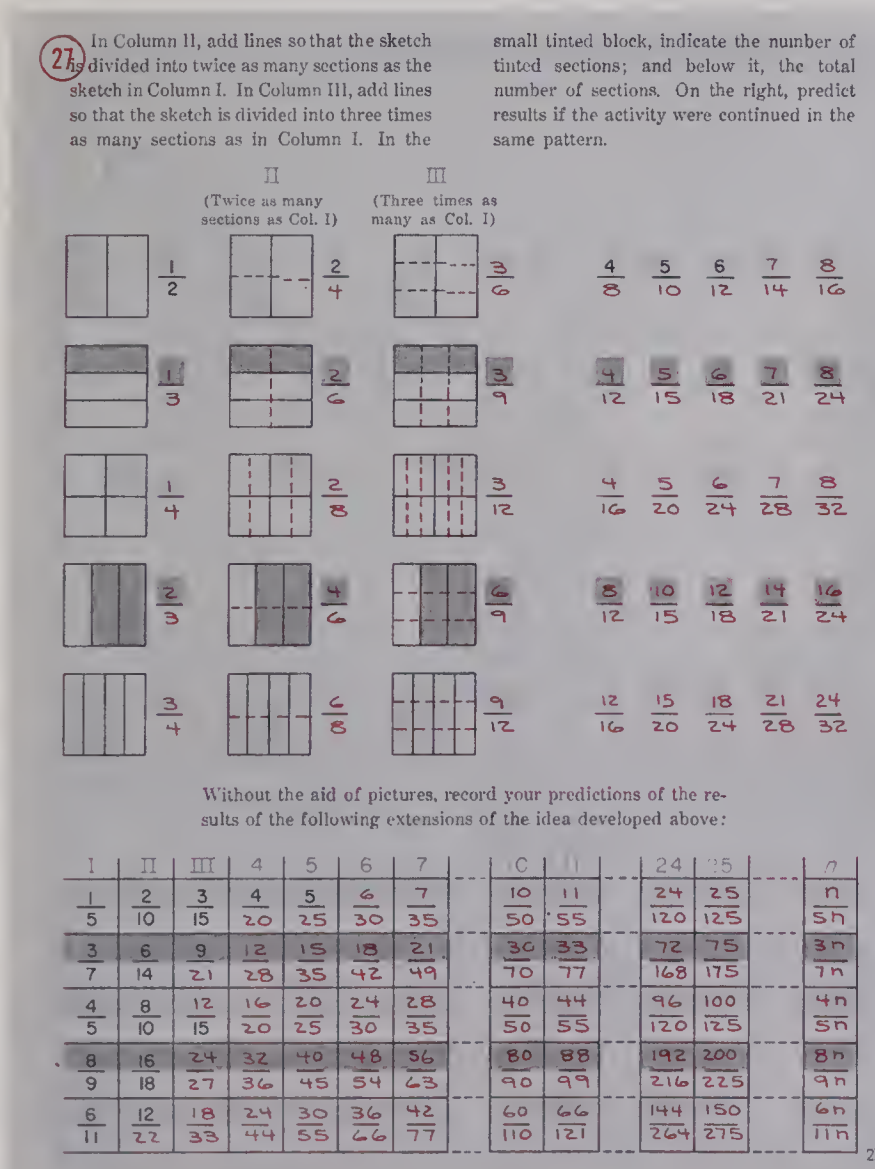
Notes to accompany *Work Text* pages 27-30

The idea that "a fraction has many names" is probably not new to many of your students. They have been reducing fractions or simplifying and finding least common denominators for some time.

$$1/2 + 1/6 = 6/12 + 2/12 = 8/12 = 2/3$$

This example involves several used of "other names" for fractions.

$$1/2 = 6/12 \quad 1/6 = 2/12 \quad 8/12 = 2/3$$


27
In Column II, add lines so that the sketch is divided into twice as many sections as the sketch in Column I. In Column III, add lines so that the sketch is divided into three times as many sections as in Column I. In the small tinted block, indicate the number of tinted sections; and below it, the total number of sections. On the right, predict results if the activity were continued in the same pattern.
II
(Twice as many sections as Col. I)
III
(Three times as many as Col. I)
Without the aid of pictures, record your predictions of the results of the following extensions of the idea developed above:
I II III 4 5 6 7 10 11 24 25 n
1/5 2/10 3/15 4/20 5/25 6/30 7/35 10/50 11/55 24/120 25/125 n/5n
3/7 6/14 9/21 12/28 15/35 18/42 21/49 30/70 33/77 72/168 75/175 3n/7n
4/5 8/10 12/15 16/20 20/25 24/30 28/35 40/50 44/55 96/120 100/125 4n/5n
8/9 16/18 24/27 32/36 40/45 48/54 56/63 80/90 88/99 192/216 200/225 8n/9n
6/11 12/22 18/33 24/44 30/55 36/66 42/77 60/110 66/121 144/264 150/275 6n/11n

And, the statement that

$$1/2 + 1/6 = 2/3$$

is asserting that a number has at least two names: One of those names is 2/3 and another is 1/2 + 1/6.

However, *Work Text* page 27 begins with five series of sketches. If the size of each suggests the idea of 1 and a certain shaded portion consequently suggests a fraction, can we introduce additional lines that change the drawing to suggest other names for the fraction? Each series of three sketches is designed to suggest such ideas as:

$$1/2 = 2/4 = 3/6$$

and the pattern begun leads us to expand to

$$3/6 = 4/8 = 5/10 = 6/12 = 7/14 = 8/16. \ldots \text{etc.},$$

indicating our confidence we could keep up that pattern indefinitely, generating new names for the fractions we have in mind.

We wonder if we could predict what the 24th member of that series would be. It begins:

1st	*2nd*	*3rd*		*24th*
1/2	2/4	3/6	....	24/48

We try another:

1st	*2nd*	*3rd*	*4th*		*24th*
3/7	6/14	9/21	12/28	....	72/168

What pattern do we notice that enables us to make that jump, confident that we would find that the 24th term would be 24/48 in one case and 72/168 in the other?

We see the second example this way:

1st	*2nd*	*3rd*	*4th*	*24th*
$\frac{1 \cdot 3}{1 \cdot 7}$	$\frac{2 \cdot 3}{2 \cdot 7}$	$\frac{3 \cdot 3}{3 \cdot 7}$	$\frac{4 \cdot 3}{4 \cdot 7}$	$\frac{24 \cdot 3}{24 \cdot 7}$

What could serve as pattern indicator in this example?

$$\frac{n \cdot 3}{n \cdot 7} \text{ or } \frac{3 \cdot n}{7 \cdot n} \text{ or } \frac{3n}{7n}$$

In any group of names for a fraction, we can always find one—or, add a name—in which there is no common whole number factor in the numerator or denominator. We say that this particular name

expresses the fraction in simplest form, or "lowest term." And, we can write a pattern indicator that will point to every other name in this family in a very simple way:

$$6/8 = 3/4 = 30/40 = 42/56 = \frac{3n}{4n}$$

In this name, *n* would be the "greatest common factor" in any name for 3/4, such as $\frac{42}{56} = \frac{3\times14}{4\times14}$ and 14 is the greatest common factor of 42 and 56.

Section A and B of page 28 develop this idea. Section C is a bit of change in pace. Marvin Hawkins comes up with an assertion that needs some testing. Ask the class whether they believe Marvin is right. Let them test his idea; let them search for an example in which his rule fails. (There is none if we exclude division by 0—and we must always.)

Some student might argue this way: If two fractions are other names for the same number, they could be written as:

$$\frac{a\times k}{b\times k} \quad \text{and} \quad \frac{a\times m}{b\times m}$$

So, Marvin says that: $(a\times k)\times(b\times m) = (a\times m)\times(b\times k)$, and $a\cdot k\cdot b\cdot m = a\cdot m\cdot b\cdot k$. That is right, but it is also getting far ahead in the game we have only begun. At this point we want to find out by trial and error whether or not Marvin has a reliable idea.

Pages 29 and 30 approach the pattern studied on pages 27 and 28 from another direction. Define "prime factors" as numbers that cannot be written as a product of whole number factors other than 1:

Primes	*Not primes*	*Because:*
2	*4*	$4 = 2\times2$
3	6	$6 = 2\times3$
5	8	$8 = 2\times2\times2$
7	9	$9 = 3\times3$
11	10	$10 = 2\times5$
	12	$12 = 2\times2\times3$

It is true that there is only one way to express each number as a product of prime factors? (1 is, of course, excluded). Order of writing the prime factors is immaterial.

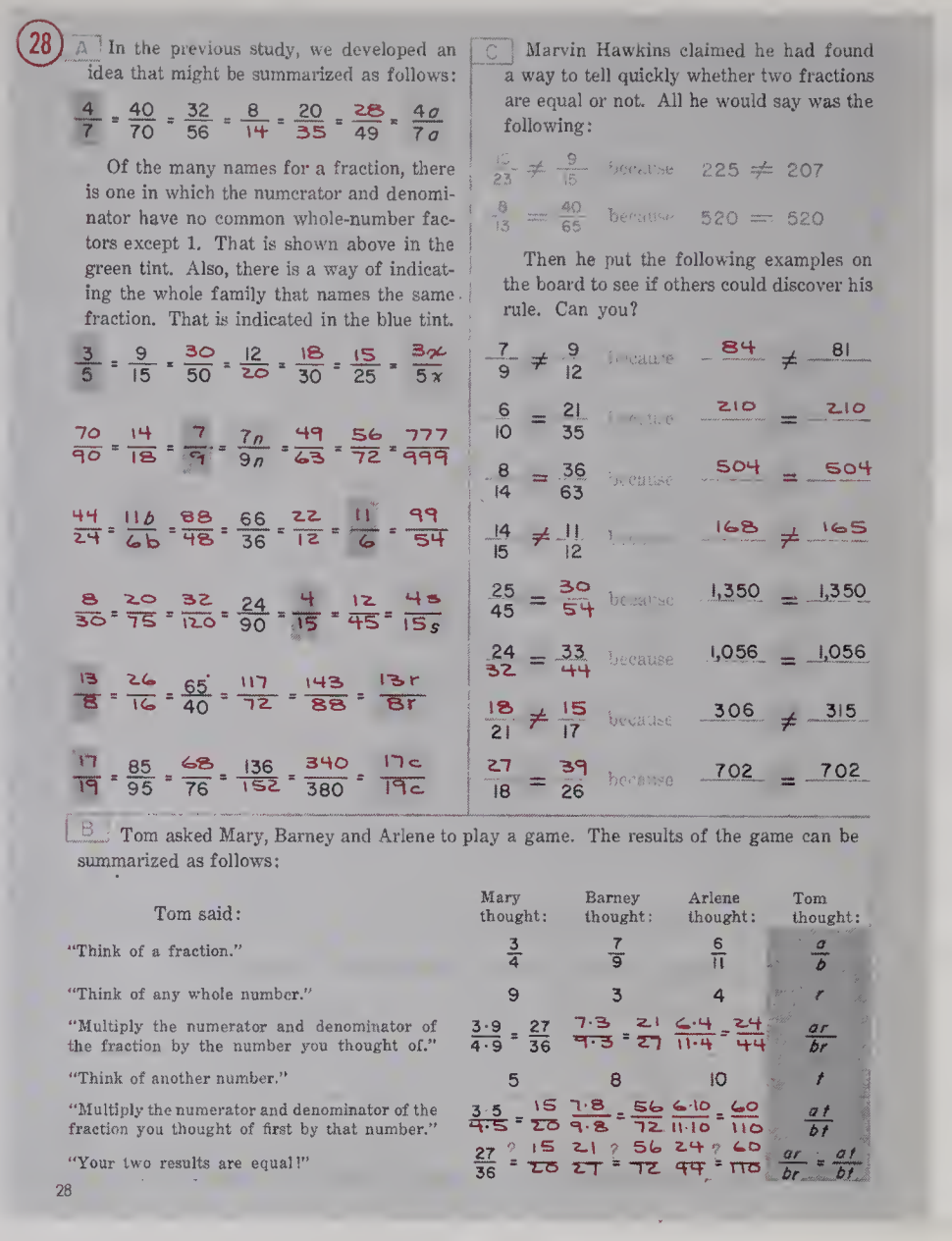
While this is true, it is not obvious without considerable analysis. Children ought to try many, many examples and convince themselves that the principle of "unique factorization" is probably a sound idea.

It follows that there must be only one way to write a fraction with the numerator and denominator each expressed as a product of prime factors. This extension of the idea is explored on page 30. In the first set of examples, common factors sometimes occur in the numerator and denominator of a fraction.

In the second group, the factors are written in an order so that the common factors are written over each other, and indicated with parentheses.

The product of the factors written inside the parentheses is the "greatest common factor" of numerator and denominator. This leads to rewriting the fraction as a pair of factors in which one factor has the greatest common factor as both numerator and denominator which is, of course, a name for 1. Now multiplication by 1 leaves a number unchanged, so the product is the same as the factor other than 1.

The result is another name for the function—a simplified name—"reduced to lowest terms." We can capture the pattern with a device that is becoming more and more familiar:


28

A In the previous study, we developed an idea that might be summarized as follows:

$\frac{4}{7} = \frac{40}{70} = \frac{32}{56} = \frac{8}{14} = \frac{20}{35} = \frac{28}{49} = \frac{4a}{7a}$

Of the many names for a fraction, there is one in which the numerator and denominator have no common whole-number factors except 1. That is shown above in the green tint. Also, there is a way of indicating the whole family that names the same fraction. That is indicated in the blue tint.

$\frac{3}{5} = \frac{9}{15} = \frac{30}{50} = \frac{12}{20} = \frac{18}{30} = \frac{15}{25} = \frac{3x}{5x}$

$\frac{70}{90} = \frac{14}{18} = \frac{7}{9} = \frac{7n}{9n} = \frac{49}{63} = \frac{56}{72} = \frac{777}{999}$

$\frac{44}{24} = \frac{11b}{6b} = \frac{88}{48} = \frac{66}{36} = \frac{22}{12} = \frac{11}{6} = \frac{99}{54}$

$\frac{8}{30} = \frac{20}{75} = \frac{32}{120} = \frac{24}{90} = \frac{4}{15} = \frac{12}{45} = \frac{4s}{15s}$

$\frac{13}{8} = \frac{26}{16} = \frac{65}{40} = \frac{117}{72} = \frac{143}{88} = \frac{13r}{8r}$

$\frac{17}{19} = \frac{85}{95} = \frac{68}{76} = \frac{136}{152} = \frac{340}{380} = \frac{17c}{19c}$

C Marvin Hawkins claimed he had found a way to tell quickly whether two fractions are equal or not. All he would say was the following:

$\frac{15}{23} \neq \frac{9}{15}$ because $225 \neq 207$

$\frac{8}{13} = \frac{40}{65}$ because $520 = 520$

Then he put the following examples on the board to see if others could discover his rule. Can you?

$\frac{7}{9} \neq \frac{9}{12}$ because $84 \neq 81$

$\frac{6}{10} = \frac{21}{35}$ because $210 = 210$

$\frac{8}{14} = \frac{36}{63}$ because $504 = 504$

$\frac{14}{15} \neq \frac{11}{12}$ because $168 \neq 165$

$\frac{25}{45} = \frac{30}{54}$ because $1{,}350 = 1{,}350$

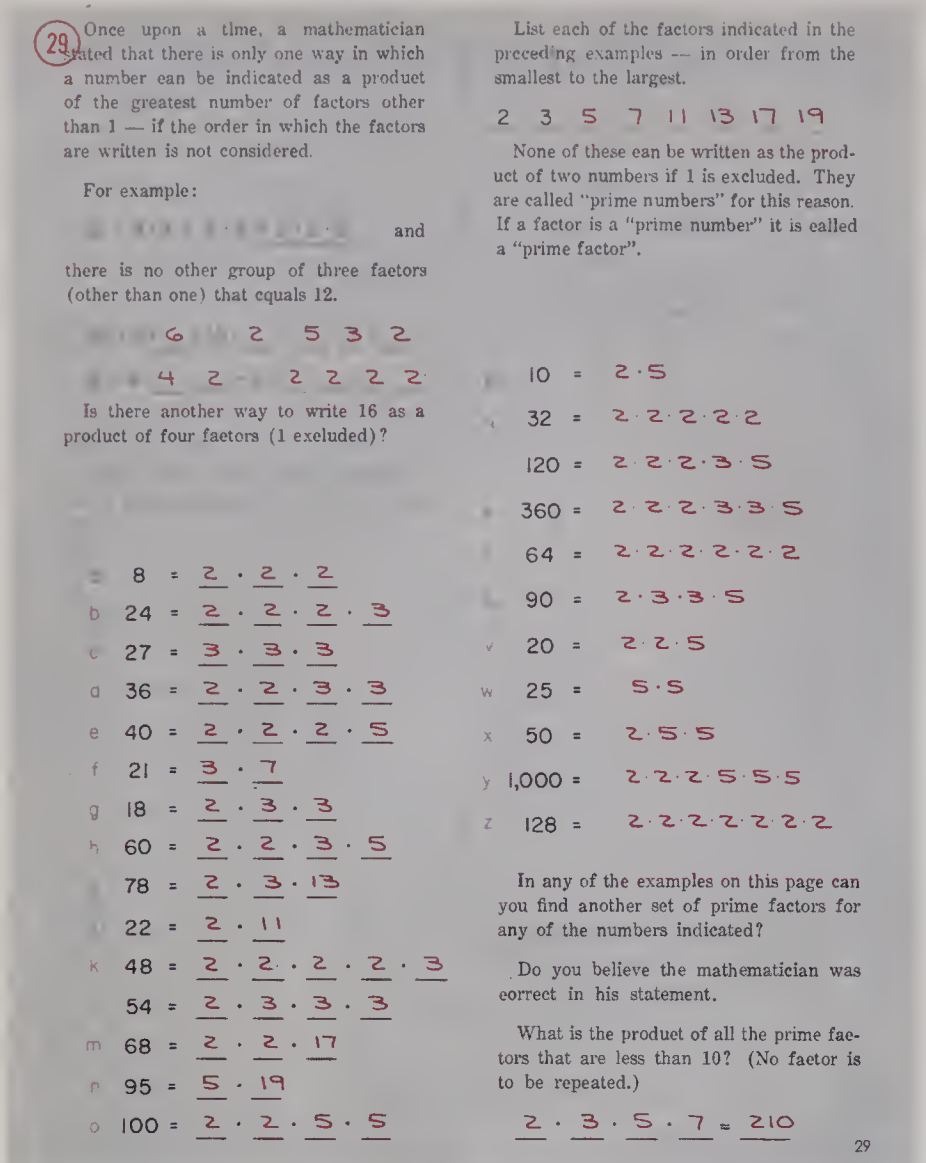
$\frac{24}{32} = \frac{33}{44}$ because $1{,}056 = 1{,}056$

$\frac{18}{21} \neq \frac{15}{17}$ because $306 \neq 315$

$\frac{27}{18} = \frac{39}{26}$ because $702 = 702$

B Tom asked Mary, Barney and Arlene to play a game. The results of the game can be summarized as follows:

Tom said:	Mary thought:	Barney thought:	Arlene thought:	Tom thought:
"Think of a fraction."	$\frac{3}{4}$	$\frac{7}{9}$	$\frac{6}{11}$	$\frac{a}{b}$
"Think of any whole number."	9	3	4	r
"Multiply the numerator and denominator of the fraction by the number you thought of."	$\frac{3\cdot9}{4\cdot9} = \frac{27}{36}$	$\frac{7\cdot3}{9\cdot3} = \frac{21}{27}$	$\frac{6\cdot4}{11\cdot4} = \frac{24}{44}$	$\frac{ar}{br}$
"Think of another number."	5	8	10	t
"Multiply the numerator and denominator of the fraction you thought of first by that number."	$\frac{3\cdot5}{4\cdot5} = \frac{15}{20}$	$\frac{7\cdot8}{9\cdot8} = \frac{56}{72}$	$\frac{6\cdot10}{11\cdot10} = \frac{60}{110}$	$\frac{at}{bt}$
"Your two results are equal!"	$\frac{27}{36} \stackrel{?}{=} \frac{15}{20}$	$\frac{21}{27} \stackrel{?}{=} \frac{56}{72}$	$\frac{24}{44} \stackrel{?}{=} \frac{60}{110}$	$\frac{ar}{br} = \frac{at}{bt}$



29

Once upon a time, a mathematician stated that there is only one way in which a number can be indicated as a product of the greatest number of factors other than 1 — if the order in which the factors are written is not considered.

For example:

[illegible] and

there is no other group of three factors (other than one) that equals 12.

[illegible] 6 2 5 3 2

[illegible] 4 2 2 2 2 2

Is there another way to write 16 as a product of four factors (1 exeluded)?

a 8 = 2 · 2 · 2

b 24 = 2 · 2 · 2 · 3

c 27 = 3 · 3 · 3

d 36 = 2 · 2 · 3 · 3

e 40 = 2 · 2 · 2 · 5

f 21 = 3 · 7

g 18 = 2 · 3 · 3

h 60 = 2 · 2 · 3 · 5

78 = 2 · 3 · 13

22 = 2 · 11

k 48 = 2 · 2 · 2 · 2 · 3

54 = 2 · 3 · 3 · 3

m 68 = 2 · 2 · 17

n 95 = 5 · 19

o 100 = 2 · 2 · 5 · 5

List each of the factors indicated in the preceding examples — in order from the smallest to the largest.

2 3 5 7 11 13 17 19

None of these can be written as the product of two numbers if 1 is excluded. They are called "prime numbers" for this reason. If a factor is a "prime number" it is called a "prime factor".

10 = 2 · 5

32 = 2 · 2 · 2 · 2 · 2

120 = 2 · 2 · 2 · 3 · 5

360 = 2 · 2 · 2 · 3 · 3 · 5

64 = 2 · 2 · 2 · 2 · 2 · 2

90 = 2 · 3 · 3 · 5

v 20 = 2 · 2 · 5

w 25 = 5 · 5

x 50 = 2 · 5 · 5

y 1,000 = 2 · 2 · 2 · 5 · 5 · 5

z 128 = 2 · 2 · 2 · 2 · 2 · 2 · 2

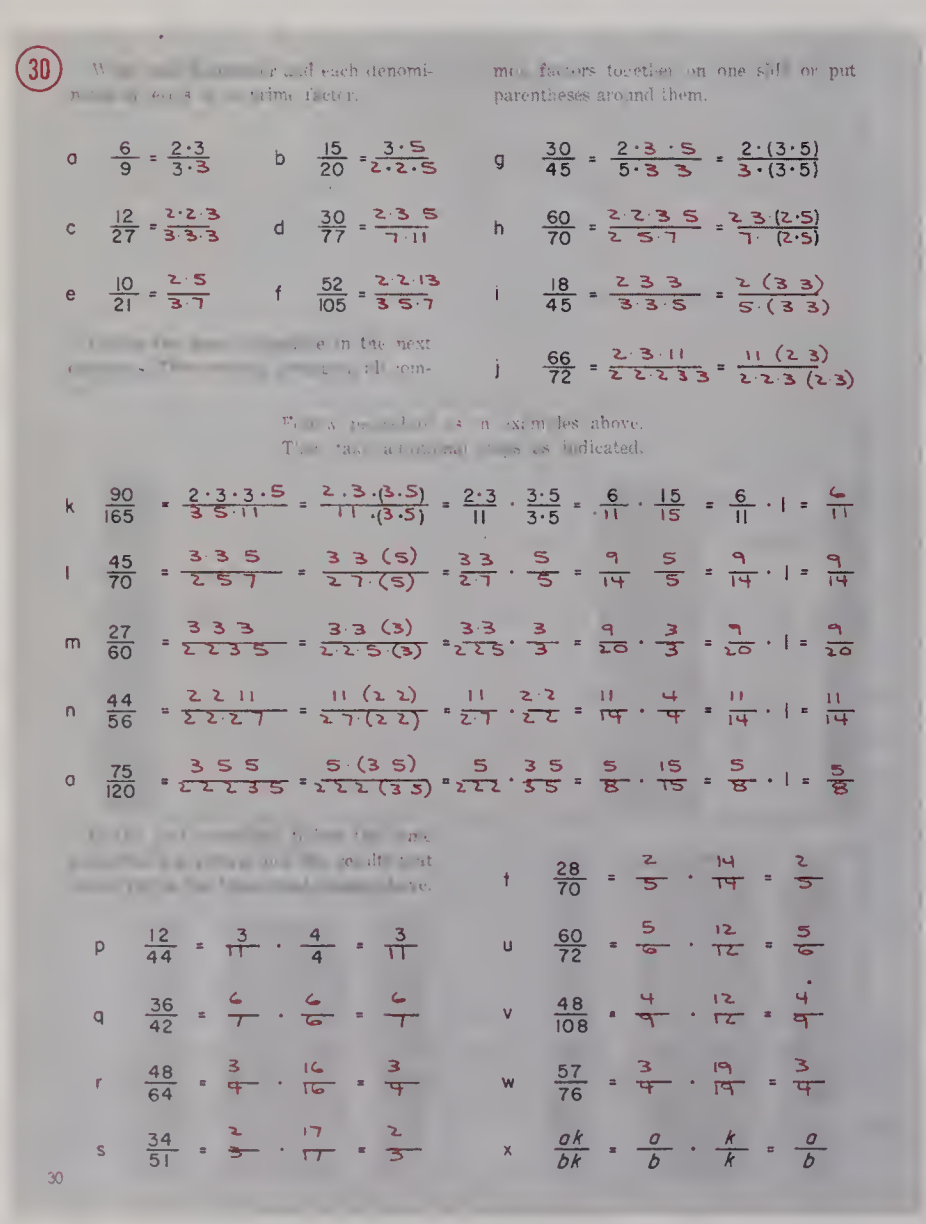
In any of the examples on this page can you find another set of prime factors for any of the numbers indicated?

Do you believe the mathematician was correct in his statement.

What is the product of all the prime factors that are less than 10? (No factor is to be repeated.)

2 · 3 · 5 · 7 = 210

29



30

[illegible] each denominator as a product of prime factors. [illegible] more factors together on one side or put parentheses around them.

a $\frac{6}{9} = \frac{2 \cdot 3}{3 \cdot 3}$

b $\frac{15}{20} = \frac{3 \cdot 5}{2 \cdot 2 \cdot 5}$

g $\frac{30}{45} = \frac{2 \cdot 3 \cdot 5}{5 \cdot 3 \ 3} = \frac{2 \cdot (3 \cdot 5)}{3 \cdot (3 \cdot 5)}$

c $\frac{12}{27} = \frac{2 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}$

d $\frac{30}{77} = \frac{2 \ 3 \ 5}{7 \cdot 11}$

h $\frac{60}{70} = \frac{2 \cdot 2 \ 3 \ 5}{2 \ 5 \cdot 7} = \frac{2 \ 3 \cdot (2 \cdot 5)}{7 \cdot (2 \cdot 5)}$

e $\frac{10}{21} = \frac{2 \cdot 5}{3 \cdot 7}$

f $\frac{52}{105} = \frac{2 \cdot 2 \cdot 13}{3 \ 5 \cdot 7}$

i $\frac{18}{45} = \frac{2 \ 3 \ 3}{3 \cdot 3 \cdot 5} = \frac{2 \ (3 \ 3)}{5 \cdot (3 \ 3)}$

[illegible] in the next [illegible] all com-

j $\frac{66}{72} = \frac{2 \cdot 3 \cdot 11}{2 \ 2 \cdot 2 \ 3 \ 3} = \frac{11 \ (2 \ 3)}{2 \cdot 2 \cdot 3 \ (2 \cdot 3)}$

[illegible] examples above. [illegible] as indicated.

k $\frac{90}{165} = \frac{2 \cdot 3 \cdot 3 \cdot 5}{3 \ 5 \cdot 11} = \frac{2 \cdot 3 \cdot (3 \cdot 5)}{11 \cdot (3 \cdot 5)} = \frac{2 \cdot 3}{11} \cdot \frac{3 \cdot 5}{3 \cdot 5} = \frac{6}{11} \cdot \frac{15}{15} = \frac{6}{11} \cdot 1 = \frac{6}{11}$

l $\frac{45}{70} = \frac{3 \ 3 \ 5}{2 \ 5 \ 7} = \frac{3 \ 3 \ (5)}{2 \ 7 \cdot (5)} = \frac{3 \ 3}{2 \ 7} \cdot \frac{5}{5} = \frac{9}{14} \ \frac{5}{5} = \frac{9}{14} \cdot 1 = \frac{9}{14}$

m $\frac{27}{60} = \frac{3 \ 3 \ 3}{2 \ 2 \ 3 \ 5} = \frac{3 \cdot 3 \ (3)}{2 \cdot 2 \cdot 5 \cdot (3)} = \frac{3 \ 3}{2 \ 2 \ 5} \cdot \frac{3}{3} = \frac{9}{20} \cdot \frac{3}{3} = \frac{9}{20} \cdot 1 = \frac{9}{20}$

n $\frac{44}{56} = \frac{2 \ 2 \ 11}{2 \cdot 2 \cdot 2 \ 7} = \frac{11 \ (2 \ 2)}{2 \ 7 \ (2 \ 2)} = \frac{11}{2 \ 7} \cdot \frac{2 \cdot 2}{2 \ 2} = \frac{11}{14} \cdot \frac{4}{4} = \frac{11}{14} \cdot 1 = \frac{11}{14}$

o $\frac{75}{120} = \frac{3 \ 5 \ 5}{2 \ 2 \ 2 \ 3 \ 5} = \frac{5 \cdot (3 \ 5)}{2 \ 2 \ 2 \ (3 \ 5)} = \frac{5}{2 \ 2 \ 2} \cdot \frac{3 \ 5}{3 \ 5} = \frac{5}{8} \cdot \frac{15}{15} = \frac{5}{8} \cdot 1 = \frac{5}{8}$

[illegible] result [illegible] above.

p $\frac{12}{44} = \frac{3}{11} \cdot \frac{4}{4} = \frac{3}{11}$

q $\frac{36}{42} = \frac{6}{7} \cdot \frac{6}{6} = \frac{6}{7}$

r $\frac{48}{64} = \frac{3}{4} \cdot \frac{16}{16} = \frac{3}{4}$

s $\frac{34}{51} = \frac{2}{3} \cdot \frac{17}{17} = \frac{2}{3}$

t $\frac{28}{70} = \frac{2}{5} \cdot \frac{14}{14} = \frac{2}{5}$

u $\frac{60}{72} = \frac{5}{6} \cdot \frac{12}{12} = \frac{5}{6}$

v $\frac{48}{108} = \frac{4}{9} \cdot \frac{12}{12} = \frac{4}{9}$

w $\frac{57}{76} = \frac{3}{4} \cdot \frac{19}{19} = \frac{3}{4}$

x $\frac{ak}{bk} = \frac{a}{b} \cdot \frac{k}{k} = \frac{a}{b}$

30

$$\frac{a \cdot k}{b \cdot k} = \frac{a}{b} \cdot \frac{k}{k} = \frac{a}{b} \cdot 1 = \frac{a}{b}$$

. . . in which a and b have no common factor.

Notes to accompany *Work Text* pages 31-32

When a product is given along with one of its factors and that given factor is a fraction, trouble develops:

$$3/4 \div 1/2 = \_\_\_\_$$

One contributing bit of confusion stems from the fact that there are so few expressions in our everyday language that parallel that mathematical question.

"How much is 3/4 of a pie divided by a half a pie?" No one says that. "How much is 75¢ divided by a half-dollar?" Nor that. Rather, people say: "How many half pieces can I get out of 3/4 of a pie?" And: "How many half-dollars can I get out of 75¢?" But 3/4 ÷ 1/2 is not a straightforward translation from the everyday language people use.

The meaning of 3/4 ÷ 1/2 becomes clearer in this expression: "How many halves are there in three-fourths?" because we parallel a familiar sentence form. If we take such liberties, we can use this kind of mathematical statement as a "headline:"

$$6 \div \frac{1}{2} = 12$$

And, if we read this as, "How many halves are there in 6?", then the mathematical sentence can be completed and appropriate stories follow.

"In 6 miles there are 12 half-mile stretches."

"I changed my 6 dollars into 12 half-dollars."

Another bit of confusion stems from a long diet of examples in arithmetic in which division led to an answer that was smaller than the dividend:

12 ÷ 3 = 4
126 ÷ 9 = 14
5 ÷ 3 = 1 2/3

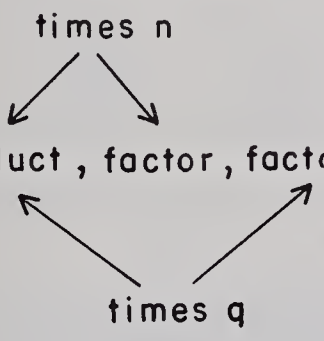
Now the pattern seems to shift:

12 ÷ 1/3 = 36
9 ÷ 3/4 = 12
1/2 ÷ 1/8 = 4

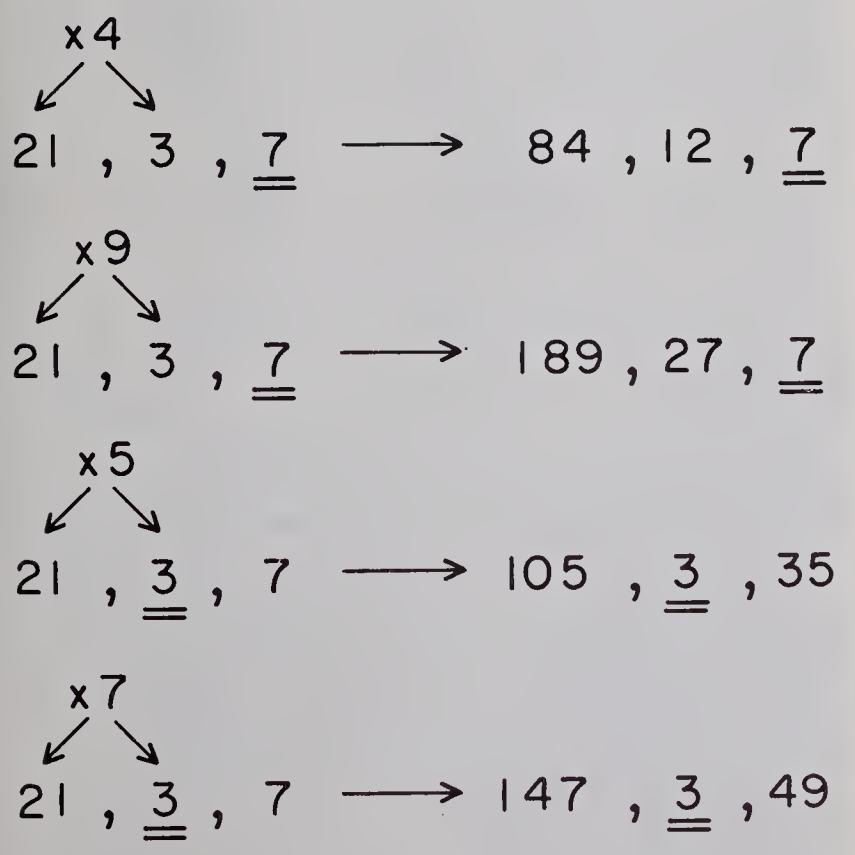
So, confusion comes from both sides: from the world of everyday events and from the many experiences with manipulation in arithmetic. Therefore, pages 31 and 32 are designed to help children: (a) see a logical method of tackling this kind of problem from a computational point of view consistent with previous developments, and (b) find examples in experience that reflect a similar pattern.

Pages 34 and 35 repeat these efforts, but along a different road. Later, we return often to consider the ideas involved when we divide by a fraction. Introduction of rules for manipulation at this point would divert attention away from the idea—and it will take some time to develop an understanding of that idea.

Mike makes an assertion. On page 31, we begin testing his arguments. We might point out Mike's idea with a sketch on the board. He is considering a situation in which we have a product and two factors:



He argues that he can think of any number, multiply the product and one of the factors by that number—and leave the other factor unchanged:



31

After much experimenting on his own, Mike presented the following argument to his class.

"Consider any true statement about a product and its factor, such as the following." (Complete each statement so that it is true.)

A $3 \cdot 4 = 12$ B $8\overline{)24}$ (3) C $\frac{42}{6} = 7$

D $\frac{1}{2} \cdot 7 = 3\frac{1}{2}$ E $3 \div \frac{1}{3} = 9$ F $12\overline{)72}$ (6)

G $\frac{4}{3} = 1\frac{1}{3}$ H $\frac{1}{2} \div \frac{1}{2} = 1$ I $\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$

"Now," Mike continued, "select any number you like. In each example, multiply the product and either of the factors by that number; and write out the results. Suppose you think of 2, and that the factor you select is indicated by the green tints above."

a $6 \cdot 4 = 24$ b $16\overline{)48}$ (3) c $\frac{84}{12} = 7$

d $1 \cdot 7 = 7$ e $6 \div \frac{1}{3} = 18$ f $24\overline{)144}$ (6)

g $\frac{8}{6} = 1\frac{1}{3}$ h $1 \div 1 = 1$ i $\frac{2}{3} \cdot \frac{4}{3} = \frac{8}{9}$

"If you had selected the other factor and multiplied it and the product by 2, the results are:"

$3 \cdot 8 = 24$ $8\overline{)48}$ (6) $\frac{84}{6} = 14$

$\frac{1}{2} \cdot 14 = 7$ $6 \div \frac{2}{3} = 9$ $12\overline{)144}$ (12)

$\frac{8}{3} = 2\frac{2}{3}$ $1 \div \frac{1}{2} = 2$ $\frac{4}{3} \cdot \frac{2}{3} = \frac{8}{9}$

"But there's nothing particular about choosing 2 for the experiment," Mike explained, "so I tried 5, multiplying the product and the factor indicated by the green tint, each by 5. My results were as follows".

$15 \cdot 4 = 60$	$40\overline{)120}$ (3)	$\frac{210}{30} = 7$
$\frac{5}{2} \cdot 7 = 17\frac{1}{2}$	$15 \div \frac{1}{3} = 45$	$60\overline{)360}$ (6)
$\frac{20}{15} = 1\frac{1}{3}$	$\frac{5}{2} \div \frac{5}{2} = 1$	$\frac{2}{3} \cdot \frac{10}{3} = \frac{20}{9}$

"Next," Mike explained, "I thought of a fraction — $\frac{1}{2}$ — and extended the experiment, multiplying the product and the factor indicated by the green tint by $\frac{1}{2}$."

a $\frac{3}{2} \cdot 4 = 6$	b $4\overline{)12}$ (3)	c $\frac{21}{3} = 7$
d $\frac{1}{4} \cdot 7 = \frac{7}{4}$	e $1\frac{1}{2} \div \frac{1}{3} = 4\frac{1}{2}$	f $6\overline{)36}$ (6)
g $\frac{2}{1\frac{1}{2}} = 1\frac{1}{3}$	h $\frac{1}{4} \div \frac{1}{4} = 1$	i $\frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$

"I point to the patterns I have been investigating in the following shorthand statements. I use '*n*' to indicate 'any number I think of'."

IF	IF
$a \cdot b = c$	$c \div a = b$
then	then
$an \cdot b = cn$	$cn \div an = b$
and	and
$a \cdot bn = cn$	$cn \div a = bn$

Do you agree with Mike?

32

No one could find a single example in which Mike's pattern led to any confusing results. But some wondered if the idea would be useful at all.

"Yes," Mike assured them. "One use I make is to change a division problem into an example of multiplication.

"Consider the question: How much is $6 \div \frac{1}{8}$? I think of 3 or $\frac{3}{1}$. My experiments led to the conclusion that

$$6 \div \frac{1}{3} = 6 \cdot \frac{3}{1} \div \frac{1}{3} \cdot \frac{3}{1}$$

"I chose a number — 3 or $\frac{3}{1}$ — with a purpose in mind. I knew that $\frac{1}{3} \cdot \frac{3}{1}$ equals 1. Now, I know that

$$6 \div \frac{1}{3} = 6 \cdot 3 \div 1 = 6 \cdot 3 = 18$$

"The trick here depends on the fact that any number divided by 1 is unchanged:

$3 \div 1 = 3$, $\frac{1}{2} \div 1 = \frac{1}{2}$, $\frac{2}{3} \div 1 = \frac{2}{3}$, $8.16 \div 1 = 8.16$ ... etc.

"Next consider a more complicated example such as $\frac{7}{8} \div \frac{2}{3} = ?$ The number I think of is $\frac{3}{2}$. The results are

$$\frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \cdot \frac{3}{2} \div \frac{2}{3} \cdot \frac{3}{2} = \frac{7}{8} \cdot \frac{3}{2} \div 1 = \frac{21}{16} = 1\frac{5}{16}$$

"No matter what number the known factor is, I think of a number which, when multiplied by the known factors, gives 1 as the result."

$7 \cdot \frac{1}{7} = 1$	$11 \cdot \frac{1}{11} = 1$	$\frac{1}{2} \cdot \frac{2}{1} = 1$
$\frac{2}{3} \cdot \frac{3}{2} = 1$	$\frac{5}{4} \cdot \frac{4}{5} = 1$	$1\frac{1}{4} \cdot \frac{4}{5} = 1$
$87 \cdot \frac{1}{87} = 1$	$\frac{7}{31} \cdot \frac{31}{7} = 1$	$2\frac{1}{2} \cdot \frac{2}{5} = 1$

"The pattern I follow can be indicated in shorthand. Since every number I know can be written in the form of a fraction, $\frac{a}{b}$ stands for any number I think of. Then

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ab} = 1$$

"Here are more examples in which I use this idea:"

a. $\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \cdot \frac{2}{1} \div \frac{1}{2} \cdot \frac{2}{1} = \frac{6}{4} = 1\frac{1}{2}$

b. $1\frac{1}{8} \div \frac{3}{8} = \frac{9}{8} \cdot \frac{8}{3} \div \frac{3}{8} \cdot \frac{8}{3} = \frac{72}{24} = 3$

c. $2 \div 3 = \frac{2}{1} \cdot \frac{1}{3} \div \frac{3}{1} \cdot \frac{1}{3} = \frac{2}{3}$

d. $\frac{1}{2} \div 2 = \frac{1}{2} \cdot \frac{1}{2} \div \frac{2}{1} \cdot \frac{1}{2} = \frac{1}{4}$

e. $\frac{3}{8} \div \frac{3}{4} = \frac{3}{8} \cdot \frac{4}{3} \div \frac{3}{4} \cdot \frac{4}{3} = \frac{12}{24} = \frac{1}{2}$

Are these results reasonable?

Think of each example as a headline. Can you make up several stories that fit each headline?

Write headlines for the following:

$\frac{2}{3} \div \frac{1}{12} = 8$

Mary cut $\frac{2}{3}$ of a yard of ribbon into pieces $\frac{1}{12}$ of a yard long. She had 8 pieces.

$3\frac{1}{2} \div \frac{1}{4} = 14$

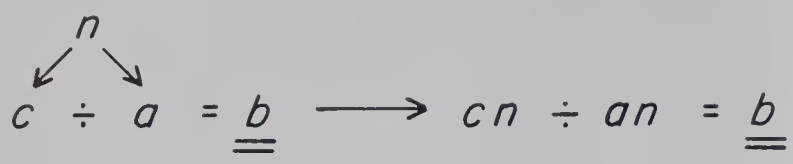
How many quarters are there in three and one-half dollars? 14

$6\frac{1}{2} \div \frac{2}{3} = 9\frac{3}{4}$

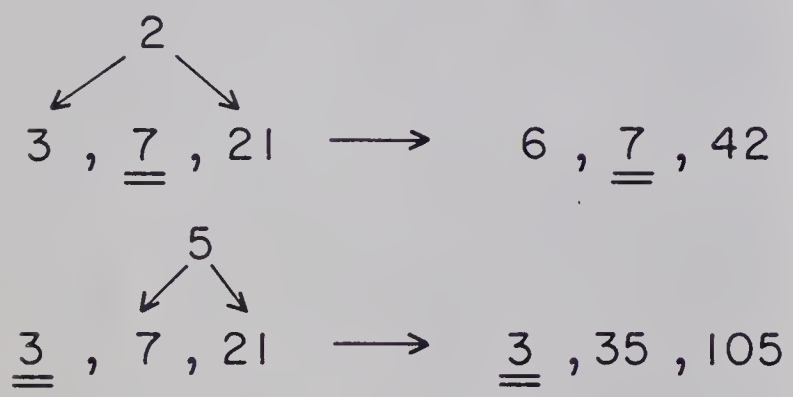
A cricket could jump for two-thirds of a mile before getting tired. How many stretches of $\frac{2}{3}$ miles would he hop in a straight line to get $6\frac{1}{2}$ miles from home?

$9\frac{3}{4}$ stretches

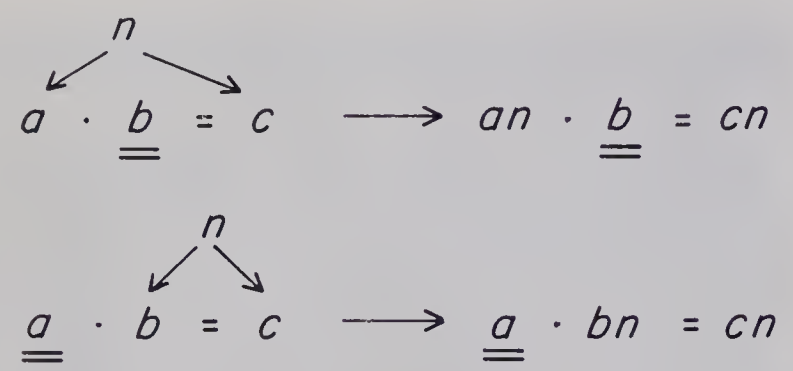
Then he suggests pattern "indicators" that might look like this in the beginning:



And, of course, the factors and product can be rearranged.



"Pattern indicators" might look like this in the beginning:



Now, this is all very interesting, but of what use is it?

Mike has an answer on page 32. Let him explain his application—without teacher help (as much as possible). After he has demonstrated his point, we turn to the key question: "Are his results reasonable?"

"Consider each of his results as a 'headline.' Can you write a sensible story to fit each?"

$$1\frac{1}{8} \div \frac{3}{8} = 3$$

"If 1 1/8'' is cut up into pieces 3/8'' long, there will be 3 pieces."

"Bill started to walk to his Uncle's house which was 1 1/8 miles away. After each 3/8 of a mile, he stopped to rest. So, he must have stopped twice." Is this a fitting story? Do the ideas in the headline and the idea in the story reflect one another? They do.

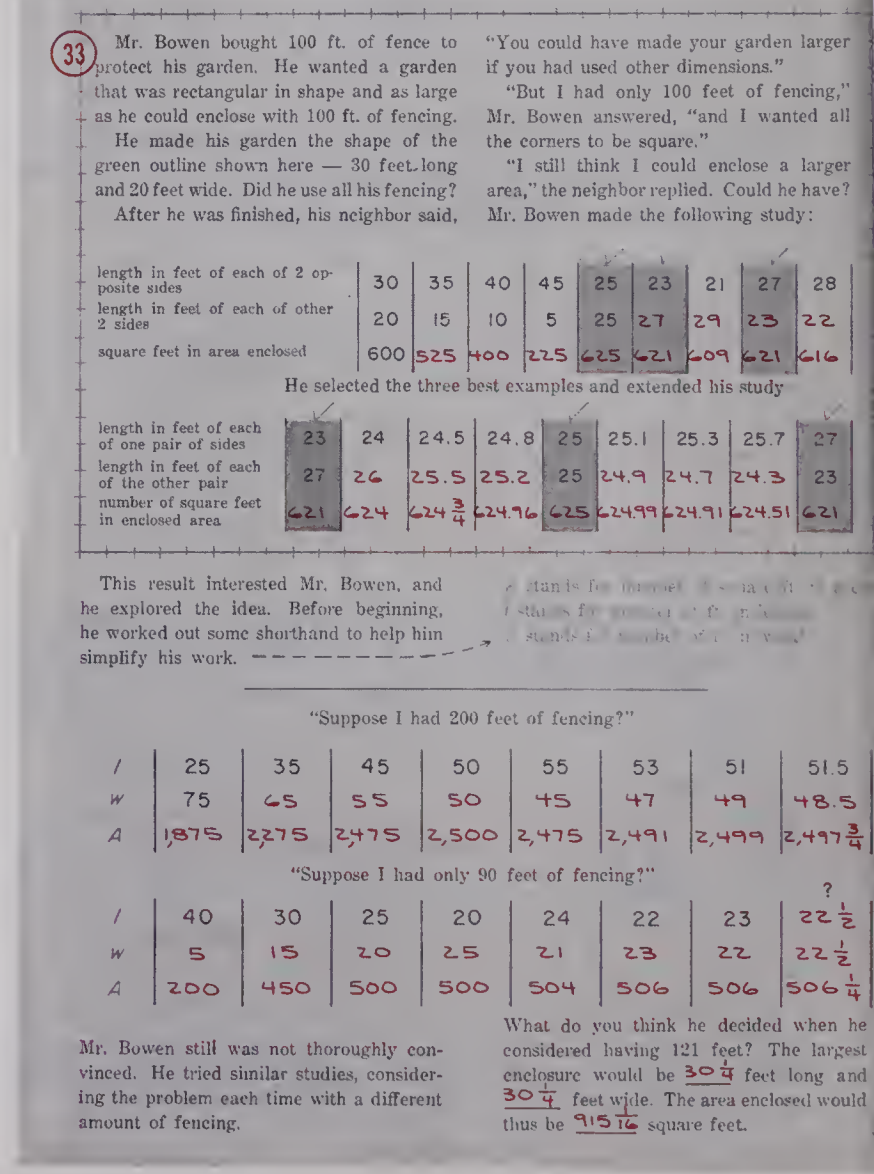
Notes to accompany *Work Text* page 33

After working out the results Mr. Bowen considers, the conclusion is reached that he ought to use his 100 feet of fencing to enclose a 25 foot by 25 foot square if he wants the largest area enclosed by that amount of fencing.

Ask the question: "How can you be sure this is true?" Point out that we are jumping to a conclusion based really on only three computations:

length in ft.		width in ft.		area in sq. ft.
24.8	×	25.2	=	624.96
25	×	25	=	625
25.1	×	24.9	=	624.99

"We haven't tried 24.99 × 25.01 = ____." (Try it.) "Would someone volunteer to try 24.999 × 25.001?"



33

Mr. Bowen bought 100 ft. of fence to protect his garden. He wanted a garden that was rectangular in shape and as large as he could enclose with 100 ft. of fencing.

He made his garden the shape of the green outline shown here — 30 feet long and 20 feet wide. Did he use all his fencing?

After he was finished, his neighbor said, "You could have made your garden larger if you had used other dimensions."

"But I had only 100 feet of fencing," Mr. Bowen answered, "and I wanted all the corners to be square."

"I still think I could enclose a larger area," the neighbor replied. Could he have? Mr. Bowen made the following study:

length in feet of each of 2 opposite sides	30	35	40	45	25	23	21	27	28
length in feet of each of other 2 sides	20	15	10	5	25	27	29	23	22
square feet in area enclosed	600	525	400	225	625	621	609	621	616

He selected the three best examples and extended his study

length in feet of each of one pair of sides	23	24	24.5	24.8	25	25.1	25.3	25.7	27
length in feet of each of the other pair	27	26	25.5	25.2	25	24.9	24.7	24.3	23
number of square feet in enclosed area	621	624	624 3/4	624.96	625	624.99	624.91	624.51	621

This result interested Mr. Bowen, and he explored the idea. Before beginning, he worked out some shorthand to help him simplify his work.

"Suppose I had 200 feet of fencing?"

l	25	35	45	50	55	53	51	51.5
w	75	65	55	50	45	47	49	48.5
A	1,875	2,275	2,475	2,500	2,475	2,491	2,499	2,497 3/4

"Suppose I had only 90 feet of fencing?"

l	40	30	25	20	24	22	23	? 22 1/2
w	5	15	20	25	21	23	22	22 1/2
A	200	450	500	500	504	506	506	506 1/4

Mr. Bowen still was not thoroughly convinced. He tried similar studies, considering the problem each time with a different amount of fencing.

What do you think he decided when he considered having 121 feet? The largest enclosure would be 30 1/4 feet long and 30 1/4 feet wide. The area enclosed would thus be 915 1/16 square feet.

(Someone will.) "Let's try one more. What shall it be?" Perhaps 24.999999 × 25.000001 = _____.

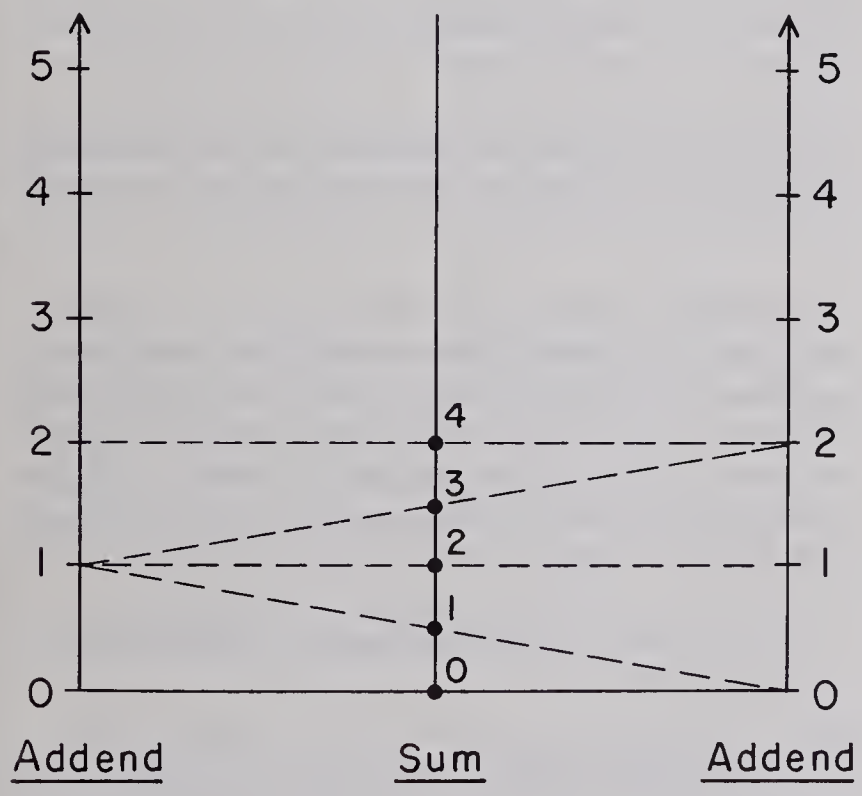
"What is the pattern in our results?" The closer we get to 25 × 25, the larger the results; and, the closer the result approaches 625.

"Could you always get closer than the last attempt anyone might make?" (This is a very important idea that will occur later when we try to find the square root of 2. It will grow to become one of the most basic mathematical ideas. But, at this stage, the discussion need go no further.)

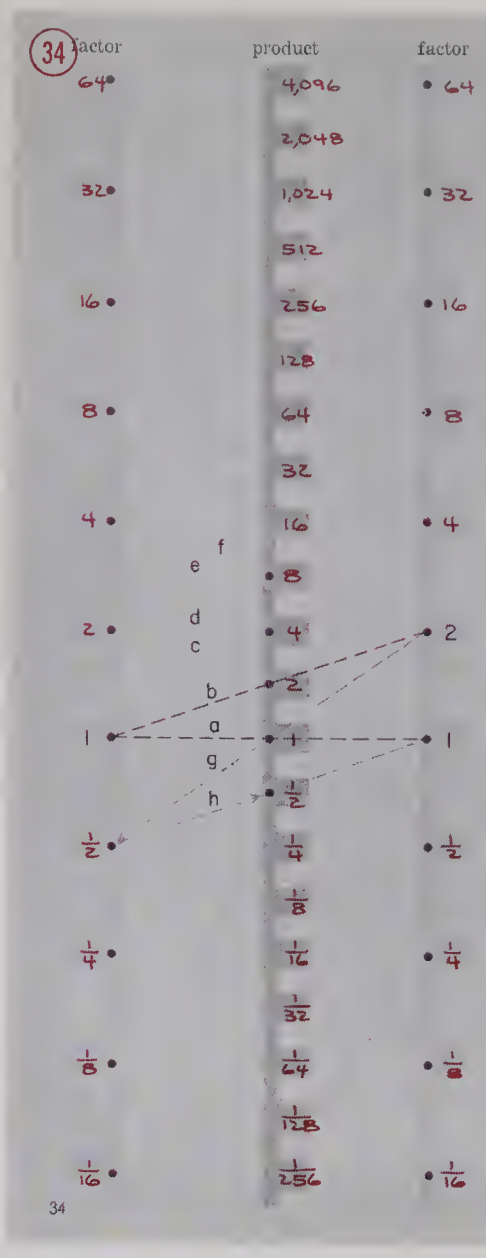
Notes to accompany *Work Text* pages 34-35

Pages 34 and 35 come back for another look at such expressions as: 1/4 ÷ 1/2 = 2 and 2 ÷ 1/4 = 8.

In earlier levels of the *Math Workshop,* the idea of "adding on a ladder" was introduced. It has some resemblance to the multiplying machines on pages 16 and 17. Here is a chunk of the ladder you might put on the board:



Connect 1 and 0 with a straightedge. Mark the point it crosses the center line and label that point as the sum of 1 and 0: 1 + 0 = 1. Connect 1 and 1– and label the point on the center line as the sum of 1 and 1: 1 + 1 = 2. Keep locating and naming points on the center line. Soon you have an "adding and subtracting" machine. (It can be expanded "downwards," to include negative as well as positive numbers.)



You can build a very limited but useful multiplying machine out of the diagram on the left.

Only three labels are given: 1, 1, and 2.

The plan is that the numbers noted on the blue scales are factors. Where a line connecting two factors cross the green line, draw a dot and label it to indicate the product.

Lines a and b at the left ask that you complete the following statements:

a. 1 = 1 ÷ 1

b. 1 = 2 ÷ 2

The point at which line a crosses the green line will be labeled 1; the point at which b crosses will be labeled 2.

Other lines on the left ask that you complete the following statements:

c. 2 = 2 ÷ 1

d. 2 = 4 ÷ 2

e. 4 = 4 ÷ 1

f. 4 = 8 ÷ 2

Keep following the same method to find and label all points on the factor lines and the product lines, above line a.

Line g at the left asks that you complete the following statement — to find the missing factor as the label for the first dot on the left below the one labeled 1.

g. 1/2 = 1 ÷ 2

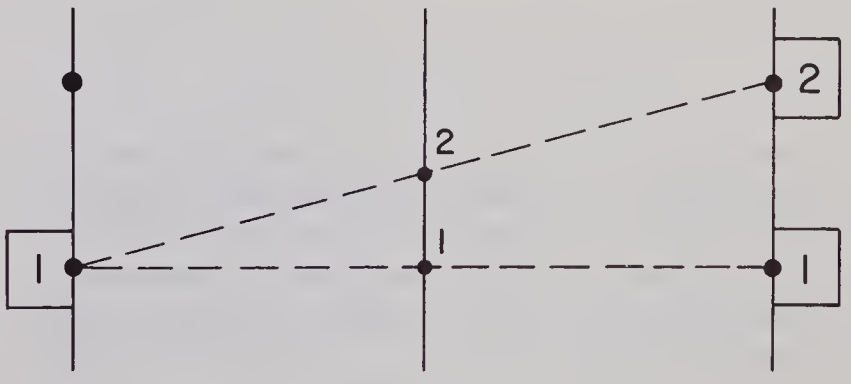
Now line h at the left asks the following:

h. 1/2 = 1/2 ÷ 1

Keep following the same method to find labels for points in the lower part of the diagram.

The same "ladder" diagram is less adaptable to being developed into a useful "multiplying and dividing" machine, but the effort is partially successful—and certainly revealing. We begin by arbitrarily assigning *1* as labels for a pair of opposite points on the ladder. When 1 and 1 are considered as factors, they point to a particular product—1. So, we attach the label 1 to the point at which a connecting line from the already labeled points on outside lines cross the center line.

Next, we arbitrarily label a point on one of the outside lines as 2, and consider a connecting line from this point to 1 on the opposite side. Where this line crosses the center line, we attach a label: 2— because 1 × 2 = 2. Now our machine looks like this:



We draw a line connecting 1 on the right line to 2 on the center line, and extend it to meet the left

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Use the multiplying machine you have built to fill in the products in the green space in the diagram at the right.

Select a factor from each of the group of numbers indicated in the blue spaces. Find them on the blue lines on the previous page. Connect them with a straight-edge. Note the indicated product.

X	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
8	64	32	16	8	4	2	1	$\frac{1}{2}$
4	32	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
2	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
1	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
$\frac{1}{2}$	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
$\frac{1}{4}$	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
$\frac{1}{8}$	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
$\frac{1}{16}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$

Find the missing factor.

A. $4 \div 8 = \frac{1}{2}$

B. $2 \div 16 = \frac{1}{8}$

C. $2 \div \frac{1}{2} = 4$

D. $2 \div \frac{1}{4} = 8$

E. $1 \div \frac{1}{2} = 2$

F. $1 \div \frac{1}{4} = 4$

G. $\frac{1}{2} \div \frac{1}{2} = 1$

H. $\frac{1}{2} \div \frac{1}{4} = 2$

I. $\frac{1}{4} \div \frac{1}{2} = \frac{1}{2}$

J. $\frac{1}{8} \div \frac{1}{16} = 2$

K. $\frac{1}{8} \div \frac{1}{8} = 1$

L. $\frac{1}{8} \div \frac{1}{4} = \frac{1}{2}$

M. $\frac{1}{8} \div \frac{1}{2} = \frac{1}{4}$

N. $\frac{1}{64} \div \frac{1}{16} = \frac{1}{4}$

O. $\frac{1}{128} \div \frac{1}{8} = \frac{1}{16}$

Write appropriate examples that fit the headlines below:

P. $8 \div \frac{1}{2} = 16$ In $8.00 there are 16 half-dollars.

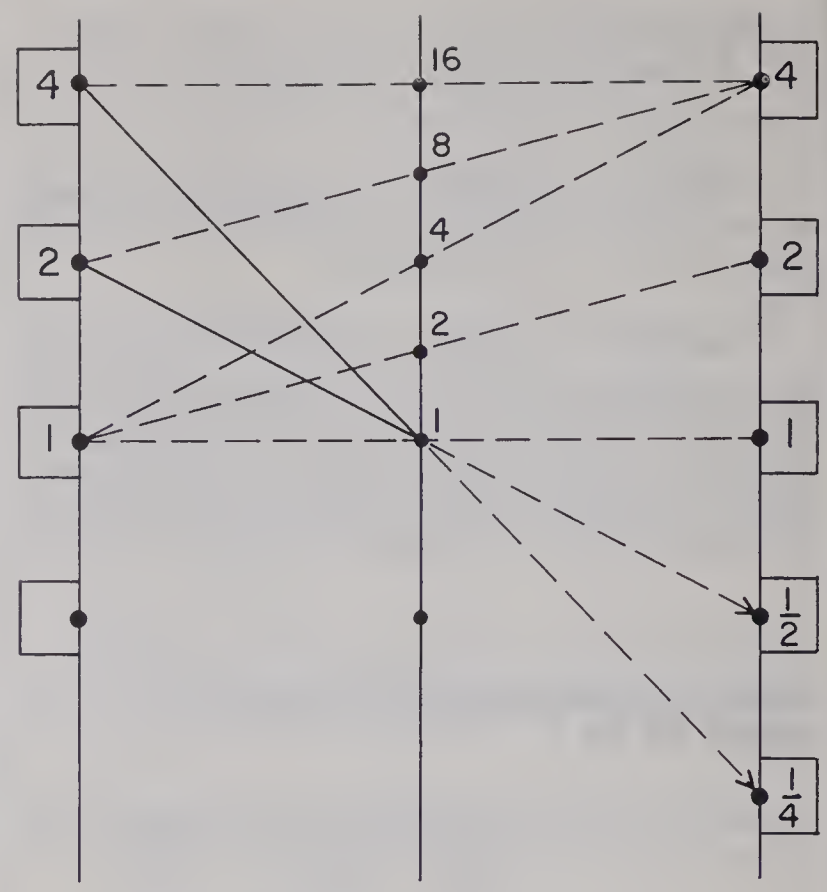
Q. $\frac{1}{2} \div \frac{1}{4} = 2$ We had a half-gallon of ice cream for 2 parties. That would be a quarter of a gallon (1 quart) for each party.

R. $\frac{1}{2} \div 4 = \frac{1}{8}$ A stretch of pavement was layed by four companies Each did $\frac{1}{8}$ of a mile—or $\frac{1}{2}$ mile altogether.

S. $\frac{1}{4} \div 2 = \frac{1}{8}$ A quarter of a pie was divided between 2 boys. Each got $\frac{1}{8}$ of a pie

T. $\frac{1}{16} \div \frac{1}{8} = \frac{1}{2}$ In $\frac{1}{16}$ of an inch there isn't $\frac{1}{8}$ of an inch— only a $\frac{1}{2}$ of $\frac{1}{8}$ of an inch.

35



Connect 2 on either outside line with 1 on the center line, and extend it as indicated above. The new point must be labeled 1/2, because $1 \div 2 = 1/2$ and $2 \times 1/2 = 1$. Connect 4 on either outside point with 1 on the center line, and extend it . . . and we label the new point 1/4 because $1 \div 4 = 1/4$ and $4 \times 1/4 = 1$.

In similar fashion, we find labels for all the points shown on all three lines. We thus produce a limited "multiplying and dividing" machine.

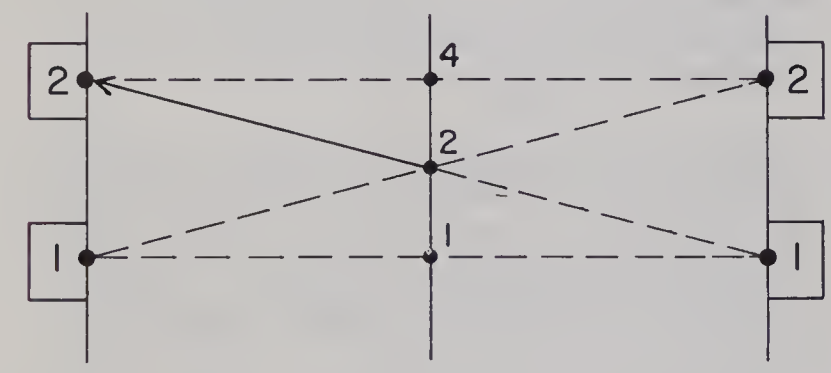
On *Work Text* page 35, we find many results by using our machine. All the results that make sense, such as $2 \times 4 = 8$ and $16 \div 2 = 8$, are familiar and we have some confidence in the reliability of the patterns used to construct the machine. A chart showing the results of 64 operations on the machine has a flawless pattern.

Now, among our results from the machine operations, we find:

$$8 \div 1/2 = 16$$

"Does this make sense? Let's see if we can make up stories that would fit this headline."

Someone might expand on the idea of the headline in this way: "I took $8 to the bank to get some change. If I asked for change in dollars, I would get $8. If I asked for change in quarters, I would get 32 quarters. But I decided to get all the change in half-dollars. I got 16 half-dollars."



To move on up, we connect 1 on the left line and 4 on the center line, and then extend that to the line on the right . . . to find a point we label 4, because $4 \div 1 = 4$ and $1 \times 4 = 4$. This pattern of construction is followed step by step to find labels for more and more points.

We turn our attention to a similar point labeling process below the points we originally labeled 1:

"In 8 hours, the big hand makes 16 half-revolutions." And, someone might add: "32 quarter-revolutions, or 8 full revolutions, or 4 times around twice." (The last expression is clumsy, but the idea is there.)

Try another result as a "headline:"

$$1/2 \div 1/4 = 1/8 \quad .$$

"If a class period is 40 minutes and half of it is divided equally to hear 4 reports, each report will be allowed 5 minutes—or, 1/8 of the 40 minute period."

As the stories are offered, one result after another that resulted from operation of our machine seems to reflect patterns we find in the world around us. The activity develops a deeper insight into "division of fractions by fractions"—or finding the missing factor when the given product and factor are both fractions.

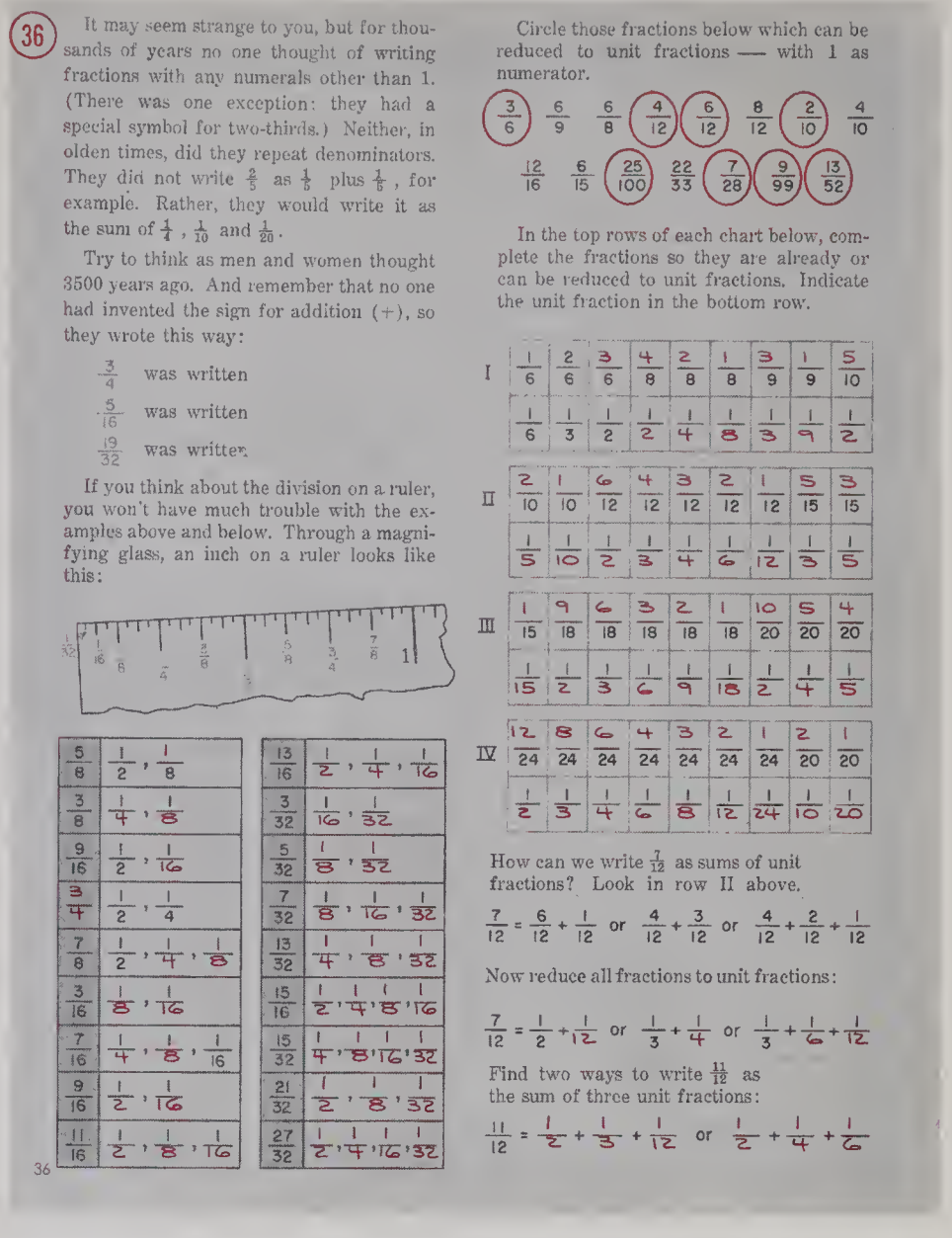
Notes to accompany *Work Text* pages 36-38

Three or four thousand years ago, men knew a lot about working with the ideas of fractions. Whole numbers had already become inadequate to express the idea that if a bushel of wheat was worth four chickens, how much wheat ought to be exchanged for one chicken?

But they knew much less about writing down names for parts of a whole. They could not write 3/4 because it didn't have any meaning to them. They saw that if 1 bushel of wheat was worth 4 chickens, then 2 chickens ought to be worth 1/2 bushel and 1 chicken ought to be worth 1/4 bushel. So, 3 chickens ought to be worth 1/2 bushel plus 1/4 bushel. They recorded this idea in a way that was equivalent to our writing 1/2, 1/4.

Let's suppose that we have to express all fractions as a sum of "unit fractions"—that is, fractions with 1 as numerator. To get under way, we can consider the ruler. "How would we express different fractions of an inch in sums of 1/32, 1/16, 1/8, 1/4 and 1/2 inches?" The first chart on page 36 explores this problem.

Sometimes, we write fractions that are not unit fractions, but can be simplified to be unit fractions, such as: 2/4, 9/27, 3/12, 16/80 which can be simplified to 1/2, 1/3, 1/4, 1/5.



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It may seem strange to you, but for thousands of years no one thought of writing fractions with any numerals other than 1. (There was one exception: they had a special symbol for two-thirds.) Neither, in olden times, did they repeat denominators. They did not write $\frac{2}{5}$ as $\frac{1}{5}$ plus $\frac{1}{5}$, for example. Rather, they would write it as the sum of $\frac{1}{4}$, $\frac{1}{10}$ and $\frac{1}{20}$.

Try to think as men and women thought 3500 years ago. And remember that no one had invented the sign for addition (+), so they wrote this way:

$\frac{3}{4}$ was written

$\frac{5}{16}$ was written

$\frac{19}{32}$ was written

If you think about the division on a ruler, you won't have much trouble with the examples above and below. Through a magnifying glass, an inch on a ruler looks like this:

5/8	1/2, 1/8
3/8	1/4, 1/8
9/16	1/2, 1/16
3/4	1/2, 1/4
7/8	1/2, 1/4, 1/8
3/16	1/8, 1/16
7/16	1/4, 1/8, 1/16
9/16	1/2, 1/16
11/16	1/2, 1/8, 1/16

13/16	1/2, 1/4, 1/16
3/32	1/16, 1/32
5/32	1/8, 1/32
7/32	1/8, 1/16, 1/32
13/32	1/4, 1/8, 1/32
15/16	1/2, 1/4, 1/8, 1/16
15/32	1/4, 1/8, 1/16, 1/32
21/32	1/2, 1/8, 1/32
27/32	1/2, 1/4, 1/16, 1/32

Circle those fractions below which can be reduced to unit fractions — with 1 as numerator.

3/6 6/9 6/8 4/12 6/12 8/12 2/10 4/10

12/16 6/15 25/100 22/33 7/28 9/99 13/52

In the top rows of each chart below, complete the fractions so they are already or can be reduced to unit fractions. Indicate the unit fraction in the bottom row.

I	1/6	2/6	3/6	4/8	2/8	1/8	3/9	1/9	5/10
	1/6	1/3	1/2	1/2	1/4	1/8	1/3	1/9	1/2
II	2/10	1/10	6/12	4/12	3/12	2/12	1/12	5/15	3/15
	1/5	1/10	1/2	1/3	1/4	1/6	1/12	1/3	1/5
III	1/15	9/18	6/18	3/18	2/18	1/18	10/20	5/20	4/20
	1/15	1/2	1/3	1/6	1/9	1/18	1/2	1/4	1/5
IV	12/24	8/24	6/24	4/24	3/24	2/24	1/24	2/20	1/20
	1/2	1/3	1/4	1/6	1/8	1/12	1/24	1/10	1/20

How can we write $\frac{7}{12}$ as sums of unit fractions? Look in row II above.

$\frac{7}{12} = \frac{6}{12} + \frac{1}{12}$ or $\frac{4}{12} + \frac{3}{12}$ or $\frac{4}{12} + \frac{2}{12} + \frac{1}{12}$

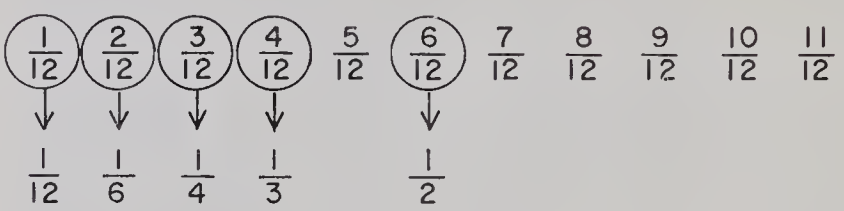
Now reduce all fractions to unit fractions:

$\frac{7}{12} = \frac{1}{2} + \frac{1}{12}$ or $\frac{1}{3} + \frac{1}{4}$ or $\frac{1}{3} + \frac{1}{6} + \frac{1}{12}$

Find two ways to write $\frac{11}{12}$ as the sum of three unit fractions:

$\frac{11}{12} = \frac{1}{2} + \frac{1}{3} + \frac{1}{12}$ or $\frac{1}{2} + \frac{1}{4} + \frac{1}{6}$

"What unit fractions less than 1 have 12 as denominator? And how can some be simplified to unit fractions?"



Using the numbers 1, 2, 3, 4, 6, in how many ways can we express 7 as a sum of those numbers—and not repeat any number in any single sum:

$7 = 1 + 6$
$7 = 4 + 3$
$7 = 4 + 2 + 1$

So, we could write 7/12 in the following ways:

$7/12 = 6/12 + 1/12$
$7/12 = 4/12 + 3/12$
$7/12 = 4/12 + 2/12 + 1/12$

The reason the numbers 1, 2, 3, 4 and 6 were chosen in this case is, because when each is a numerator of a fraction with 12 as denominator, it can be simplified to a "unit fraction." Now we can write:

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What is the largest fraction that can be expressed as the sum of 2 different unit fractions?

$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$

Is there more than one way to express that fraction as the sum of 2 different unit fractions? No

Make each of the following into fractions that are less than 1 and whose numerators are more than 1.

$\frac{2}{3}$ $\frac{8}{11}$ $\frac{4}{5}$ $\frac{9}{13}$ $\frac{10}{19}$ $\frac{12}{29}$ $\frac{5}{7}$ $\frac{14}{17}$ $\frac{15}{31}$

$\frac{16}{47}$ $\frac{17}{61}$ $\frac{18}{71}$ $\frac{20}{37}$ $\frac{30}{59}$ $\frac{40}{73}$ $\frac{50}{67}$ $\frac{25}{43}$ $\frac{49}{53}$

How many can be reduced? None

How many could be reduced if you had selected other numerators that meet the rules given above? None

Can you use any denominator less than 75 which behaves in the same way — and has not been used in one of the examples above? 23 and 41

Arrange these and the numbers used above as denominators in order of size beginning with 3, 5, etc.

3 5 7 11 13 17 19 23 29 31

37 41 43 47 53 59 61 67 71 73

How many other numbers that behave similarly can you find that are less than 100? Five

79 83 89 91 97 ___

Every whole number that is evenly divisible by only two different numbers — itself and 1 — is called a PRIME NUMBER.

Is 1 a prime number? No

Is 2 a prime number? Yes

Is there any other even number that is a prime number? No

Write fractions below in which:

a. numerators are prime numbers

$\frac{7}{12}$ $\frac{29}{58}$ $\frac{31}{60}$ $\frac{11}{22}$ $\frac{13}{17}$ $\frac{3}{12}$

b. denominators are prime numbers

$\frac{22}{11}$ $\frac{2}{13}$ $\frac{4}{19}$ $\frac{6}{31}$ $\frac{9}{41}$ $\frac{18}{61}$

c. numerators and denominators are both prime numbers.

$\frac{3}{41}$ $\frac{5}{37}$ $\frac{7}{31}$ $\frac{11}{29}$ $\frac{13}{23}$ $\frac{17}{19}$

Could you follow the directions above at c., and write a fraction that could be reduced? No

Can you write $\frac{2}{7}$ as the sum of unit fractions?

Yes $\frac{2}{7} = \frac{4}{14} = \frac{6}{21} = \frac{8}{28}$

Write out all fractions with 28 as a denominator that can be reduced to unit fractions.

	$\frac{14}{28}$	$\frac{7}{28}$	$\frac{4}{28}$	$\frac{2}{28}$	$\frac{1}{28}$
Unit fractions	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{7}$	$\frac{1}{14}$	$\frac{1}{28}$

$\frac{2}{7} = \frac{8}{28} = \frac{7}{28} + \frac{1}{28} = \frac{1}{4} + \frac{1}{28}$

$\frac{2}{7} = \frac{1}{4} + \frac{1}{28}$

We also know that $\frac{2}{7} = \frac{12}{42}$. Now complete the following so they can be reduced to unit fractions:

	$\frac{21}{42}$	$\frac{14}{42}$	$\frac{7}{42}$	$\frac{6}{42}$	$\frac{3}{42}$	$\frac{2}{42}$	$\frac{1}{42}$
Unit fractions	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{14}$	$\frac{1}{21}$	$\frac{1}{42}$

$\frac{2}{7} = \frac{12}{42} = \frac{7}{42} + \frac{3}{42} + \frac{2}{42} = \frac{1}{6} + \frac{1}{14} + \frac{1}{21}$

37

38 UNIT FRACTIONS

How many fractions less than one and with denominators less than 100 are, or can be, reduced to unit fractions? 372

$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{4}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{4}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{3}{9}$	$\frac{1}{9}$	$\frac{5}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{1}{11}$
$\frac{6}{12}$	$\frac{4}{12}$	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{1}{13}$	$\frac{7}{14}$	$\frac{2}{14}$	$\frac{1}{14}$	$\frac{5}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{8}{16}$	$\frac{4}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{1}{17}$	$\frac{9}{18}$
$\frac{6}{18}$	$\frac{3}{18}$	$\frac{2}{18}$	$\frac{1}{18}$	$\frac{1}{19}$	$\frac{10}{20}$	$\frac{5}{20}$	$\frac{4}{20}$	$\frac{2}{20}$	$\frac{1}{20}$	$\frac{7}{21}$	$\frac{3}{21}$	$\frac{1}{21}$	$\frac{11}{22}$	$\frac{2}{22}$	$\frac{1}{22}$	$\frac{1}{23}$	$\frac{12}{24}$
$\frac{8}{24}$	$\frac{6}{24}$	$\frac{4}{24}$	$\frac{3}{24}$	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{5}{25}$	$\frac{1}{25}$	$\frac{13}{26}$	$\frac{2}{26}$	$\frac{1}{26}$	$\frac{9}{27}$	$\frac{3}{27}$	$\frac{1}{27}$	$\frac{14}{28}$	$\frac{7}{28}$	$\frac{4}{28}$	$\frac{2}{28}$
$\frac{1}{28}$	$\frac{1}{29}$	$\frac{15}{30}$	$\frac{10}{30}$	$\frac{6}{30}$	$\frac{5}{30}$	$\frac{3}{30}$	$\frac{2}{30}$	$\frac{1}{30}$	$\frac{1}{31}$	$\frac{16}{32}$	$\frac{8}{32}$	$\frac{4}{32}$	$\frac{2}{32}$	$\frac{1}{32}$	$\frac{11}{33}$	$\frac{3}{33}$	$\frac{1}{33}$
$\frac{17}{34}$	$\frac{2}{34}$	$\frac{1}{34}$	$\frac{7}{35}$	$\frac{5}{35}$	$\frac{1}{35}$	$\frac{18}{36}$	$\frac{12}{36}$	$\frac{9}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{1}{37}$	$\frac{19}{38}$	$\frac{2}{38}$	$\frac{1}{38}$
$\frac{13}{39}$	$\frac{3}{39}$	$\frac{1}{39}$	$\frac{20}{40}$	$\frac{10}{40}$	$\frac{8}{40}$	$\frac{5}{40}$	$\frac{4}{40}$	$\frac{2}{40}$	$\frac{1}{40}$	$\frac{1}{41}$	$\frac{21}{42}$	$\frac{14}{42}$	$\frac{7}{42}$	$\frac{6}{42}$	$\frac{3}{42}$	$\frac{2}{42}$	$\frac{1}{42}$
$\frac{1}{43}$	$\frac{22}{44}$	$\frac{11}{44}$	$\frac{4}{44}$	$\frac{2}{44}$	$\frac{1}{44}$	$\frac{15}{45}$	$\frac{9}{45}$	$\frac{5}{45}$	$\frac{3}{45}$	$\frac{1}{45}$	$\frac{23}{46}$	$\frac{2}{46}$	$\frac{1}{46}$	$\frac{1}{47}$	$\frac{24}{48}$	$\frac{16}{48}$	$\frac{12}{48}$
$\frac{8}{48}$	$\frac{6}{48}$	$\frac{4}{48}$	$\frac{3}{48}$	$\frac{2}{48}$	$\frac{1}{48}$	$\frac{7}{49}$	$\frac{1}{49}$	$\frac{25}{50}$	$\frac{10}{50}$	$\frac{5}{50}$	$\frac{2}{50}$	$\frac{1}{50}$	$\frac{17}{51}$	$\frac{3}{51}$	$\frac{1}{51}$	$\frac{26}{52}$	$\frac{13}{52}$
$\frac{4}{52}$	$\frac{2}{52}$	$\frac{1}{52}$	$\frac{1}{53}$	$\frac{27}{54}$	$\frac{18}{54}$	$\frac{9}{54}$	$\frac{6}{54}$	$\frac{3}{54}$	$\frac{2}{54}$	$\frac{1}{54}$	$\frac{11}{55}$	$\frac{5}{55}$	$\frac{1}{55}$	$\frac{28}{56}$	$\frac{14}{56}$	$\frac{8}{56}$	$\frac{7}{56}$
$\frac{4}{56}$	$\frac{2}{56}$	$\frac{1}{56}$	$\frac{19}{57}$	$\frac{3}{57}$	$\frac{1}{57}$	$\frac{29}{58}$	$\frac{2}{58}$	$\frac{1}{58}$	$\frac{1}{59}$	$\frac{30}{60}$	$\frac{20}{60}$	$\frac{15}{60}$	$\frac{12}{60}$	$\frac{10}{60}$	$\frac{6}{60}$	$\frac{5}{60}$	$\frac{4}{60}$
$\frac{3}{60}$	$\frac{2}{60}$	$\frac{1}{60}$	$\frac{1}{61}$	$\frac{31}{62}$	$\frac{2}{62}$	$\frac{1}{62}$	$\frac{21}{63}$	$\frac{9}{63}$	$\frac{7}{63}$	$\frac{3}{63}$	$\frac{1}{63}$	$\frac{32}{64}$	$\frac{16}{64}$	$\frac{8}{64}$	$\frac{4}{64}$	$\frac{2}{64}$	$\frac{1}{64}$
$\frac{13}{65}$	$\frac{5}{65}$	$\frac{1}{65}$	$\frac{33}{66}$	$\frac{22}{66}$	$\frac{11}{66}$	$\frac{6}{66}$	$\frac{3}{66}$	$\frac{2}{66}$	$\frac{1}{66}$	$\frac{1}{67}$	$\frac{34}{68}$	$\frac{17}{68}$	$\frac{4}{68}$	$\frac{2}{68}$	$\frac{1}{68}$	$\frac{23}{69}$	$\frac{3}{69}$
$\frac{1}{69}$	$\frac{35}{70}$	$\frac{14}{70}$	$\frac{10}{70}$	$\frac{7}{70}$	$\frac{5}{70}$	$\frac{2}{70}$	$\frac{1}{70}$	$\frac{1}{71}$	$\frac{36}{72}$	$\frac{24}{72}$	$\frac{18}{72}$	$\frac{12}{72}$	$\frac{9}{72}$	$\frac{8}{72}$	$\frac{6}{72}$	$\frac{4}{72}$	$\frac{3}{72}$
$\frac{2}{72}$	$\frac{1}{72}$	$\frac{1}{73}$	$\frac{37}{74}$	$\frac{2}{74}$	$\frac{1}{74}$	$\frac{25}{75}$	$\frac{15}{75}$	$\frac{5}{75}$	$\frac{3}{75}$	$\frac{1}{75}$	$\frac{38}{76}$	$\frac{19}{76}$	$\frac{4}{76}$	$\frac{2}{76}$	$\frac{1}{76}$	$\frac{11}{77}$	$\frac{7}{77}$
$\frac{1}{77}$	$\frac{39}{78}$	$\frac{26}{78}$	$\frac{13}{78}$	$\frac{6}{78}$	$\frac{3}{78}$	$\frac{2}{78}$	$\frac{1}{78}$	$\frac{1}{79}$	$\frac{40}{80}$	$\frac{20}{80}$	$\frac{16}{80}$	$\frac{10}{80}$	$\frac{8}{80}$	$\frac{5}{80}$	$\frac{4}{80}$	$\frac{2}{80}$	$\frac{1}{80}$
$\frac{27}{81}$	$\frac{9}{81}$	$\frac{3}{81}$	$\frac{1}{81}$	$\frac{41}{82}$	$\frac{2}{82}$	$\frac{1}{82}$	$\frac{1}{83}$	$\frac{42}{84}$	$\frac{28}{84}$	$\frac{21}{84}$	$\frac{14}{84}$	$\frac{12}{84}$	$\frac{7}{84}$	$\frac{6}{84}$	$\frac{4}{84}$	$\frac{3}{84}$	$\frac{2}{84}$
$\frac{1}{84}$	$\frac{17}{85}$	$\frac{5}{85}$	$\frac{1}{85}$	$\frac{43}{86}$	$\frac{2}{86}$	$\frac{1}{86}$	$\frac{29}{87}$	$\frac{3}{87}$	$\frac{1}{87}$	$\frac{44}{88}$	$\frac{22}{88}$	$\frac{11}{88}$	$\frac{8}{88}$	$\frac{4}{88}$	$\frac{2}{88}$	$\frac{1}{88}$	$\frac{1}{89}$
$\frac{45}{90}$	$\frac{30}{90}$	$\frac{18}{90}$	$\frac{15}{90}$	$\frac{10}{90}$	$\frac{9}{90}$	$\frac{6}{90}$	$\frac{5}{90}$	$\frac{3}{90}$	$\frac{2}{90}$	$\frac{1}{90}$	$\frac{1}{91}$	$\frac{46}{92}$	$\frac{23}{92}$	$\frac{4}{92}$	$\frac{2}{92}$	$\frac{1}{92}$	$\frac{31}{93}$
$\frac{3}{93}$	$\frac{1}{93}$	$\frac{47}{94}$	$\frac{2}{94}$	$\frac{1}{94}$	$\frac{19}{95}$	$\frac{5}{95}$	$\frac{1}{95}$	$\frac{48}{96}$	$\frac{32}{96}$	$\frac{24}{96}$	$\frac{16}{96}$	$\frac{12}{96}$	$\frac{8}{96}$	$\frac{6}{96}$	$\frac{4}{96}$	$\frac{3}{96}$	$\frac{2}{96}$
$\frac{1}{96}$	$\frac{1}{97}$	$\frac{49}{98}$	$\frac{14}{98}$	$\frac{7}{98}$	$\frac{2}{98}$	$\frac{1}{98}$	$\frac{33}{99}$	$\frac{11}{99}$	$\frac{9}{99}$	$\frac{3}{99}$	$\frac{1}{99}$						

$7/12 = 1/12 + 1/2$
$7/12 = 1/4 + 1/3$
$7/12 = 1/12 + 1/6 + 1/3$

"Can 7/12 be expressed in any other ways as the sum of different unit fractions?" Yes.

Start another approach to the problem. How many fractions less than 1 and with denominators of 3×12, or 36, are, or can be, simplified to unit fractions?

1/36	2/36	3/36	4/36	6/36	9/36	12/36	18/36
↓	↓	↓	↓	↓	↓	↓	↓
1/36	1/18	1/12	1/9	1/6	1/4	1/3	1/2

(Note that, in reverse order, the denominators of the second row repeat the numerators of the top row.)

We know that: 7/12 = 21/36. Now, without repeating in any example, how many ways can 21 be expressed as a sum of 1, 2, 3, 4, 6, 9, 12 and 18?

$21 = 18 + 3$
$21 = 18 + 2 + 1$
$21 = 12 + 9$
$21 = 12 + 6 + 3$
$21 = 12 + 6 + 2 + 1$
$21 = 9 + 6 + 4 + 2$
$21 = 9 + 6 + 3 + 2 + 1$

So, the ways of writing 7/12 or 21/36 include:

$7/12 = 21/36 = 18/36 + 3/36$
$7/12 = 18/36 + 2/36 + 1/36$
$7/12 = 12/36 + 9/36$
$7/12 = 12/36 + 6/36 + 3/36$
$7/12 = 12/36 + 6/36 + 2/36 + 1/36$
$7/12 = 9/36 + 6/36 + 4/36 + 1/36$
$7/12 = 9/36 + 6/36 + 3/36 + 2/36 + 1/36$

And these can be simplified to "unit fractions," so:

$7/12 + 1/2 + 1/12$
$7/12 = 1/3 + 1/18 + 1/36$
$7/12 = 1/3 + 1/4$
$7/12 = 1/3 + 1/6 + 1/12$
$7/12 = 1/3 + 1/6 + 1/18 + 1/36$
$7/12 = 1/4 + 1/6 + 1/9 + 1/18$
$7/12 = 1/4 + 1/6 + 1/12 + 1/18 + 1/36$

Are there more? Certainly! Consider that 7/12 = 35/60 or 56/96, etc.

Some pupils may want to follow the pattern of this study by analyzing the results of considering, in order, the following:

7/12 = 7/12 7/12 = 14/24
7/12 = 21/36 7/12 = 28/48, etc.

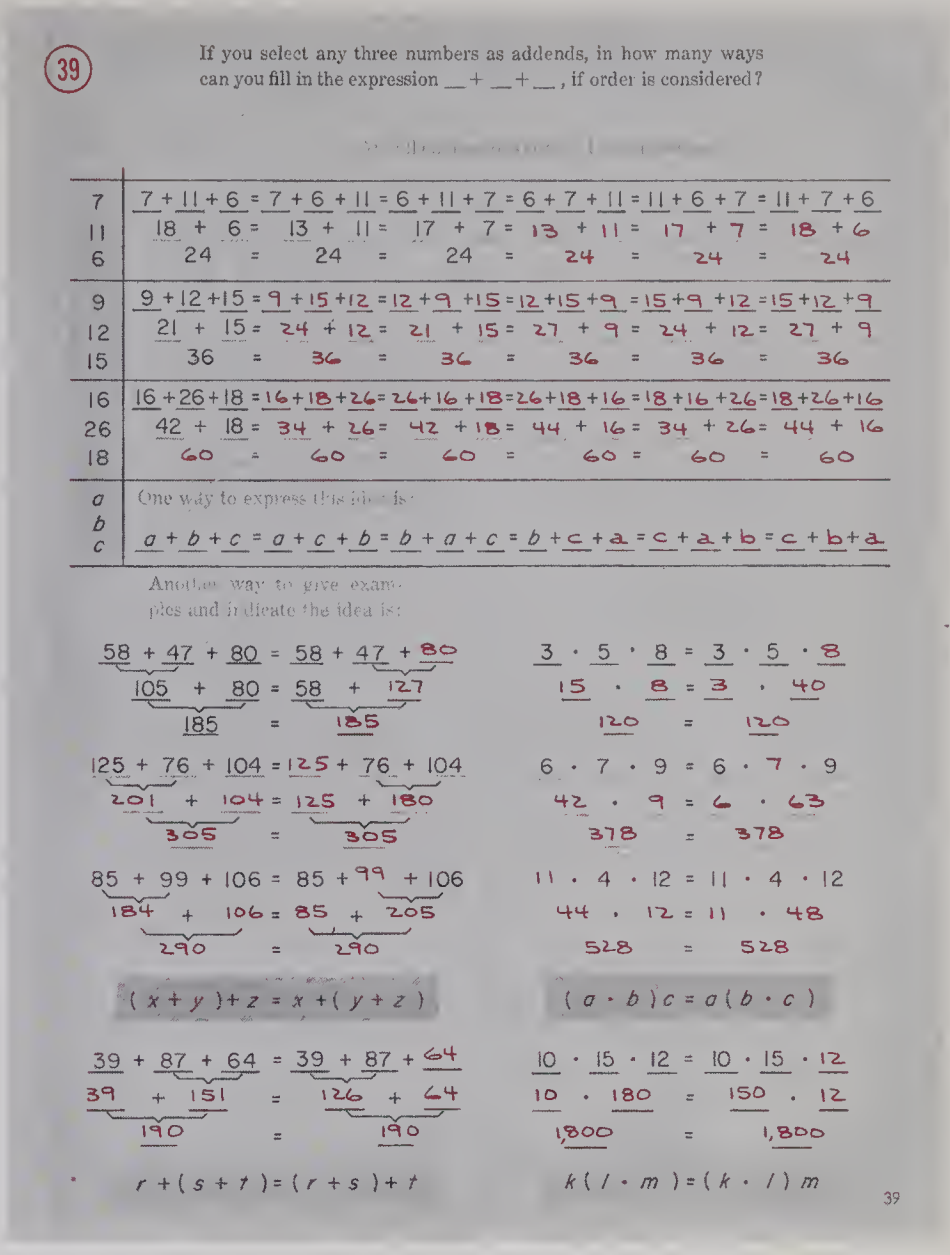
We explored the two that are underlined. In the first study, we found three ways to write 7/12 as the sum of unit fractions. In the second, we found seven ways. Did we find them all? If so, would you guess that a study of 7/12 = 14/24 would turn up with five ways? And 7/12 = 28/48 with seven ways?

We have only considered one fraction—7/12—and some of its names written as the sum of different unit fractions. All fractions can be studied in this way. Each can be written as a sum of unit fractions in endless ways . . . with as many addends as desired. Each study can be a completely open-ended activity.

Finally, on *Work Text* page 38, we study another question about unit fractions. How many fractions are there that are less than 1 and have denominators less than 100—which are, or can be, simplified to unit fractions?

"Are there any omitted on page 38?"

The answer to this question can be found only if each possible denominator is approached much as we have considered 12 and 36 above.



Each family shown on page 38 in which there is only one member, is written over a blue tint. What can be said about the denominators of each fraction written in the blue tint? They have no other factors than themselves and 1. They are, of course, the prime numbers. Do they include all prime numbers less than 100? Yes.

And throughout this study we have had a surprising amount of experience in simplifying or reducing fractions.

Notes to accompany *Work Text* pages 39-43

Page 39 initiates another pattern hunt—in search, once again, of patterns that are so important mathematicians have given them names: the "associative principle of addition" and the "associative principle of multiplication." In more everyday language, you can name these ideas an important part of the "in-any-order" rules for addition and for multiplication. But save all discussions until the children have seen a pattern worth naming.

We approach this activity from a side road: "In how many ways can we use 7, 11 and 6 as addends?"

7 + 11 + 6 11 + 7 + 6
7 + 6 + 11 6 + 7 + 11
11 + 6 + 7 6 + 11 + 7

Arranged in this way, we can be quite sure we have listed all the combinations. Now we limit ourselves to finding a sum of only 2 addends at a time:

7 + 11 + 6 =
↓
18 + 6 = 24

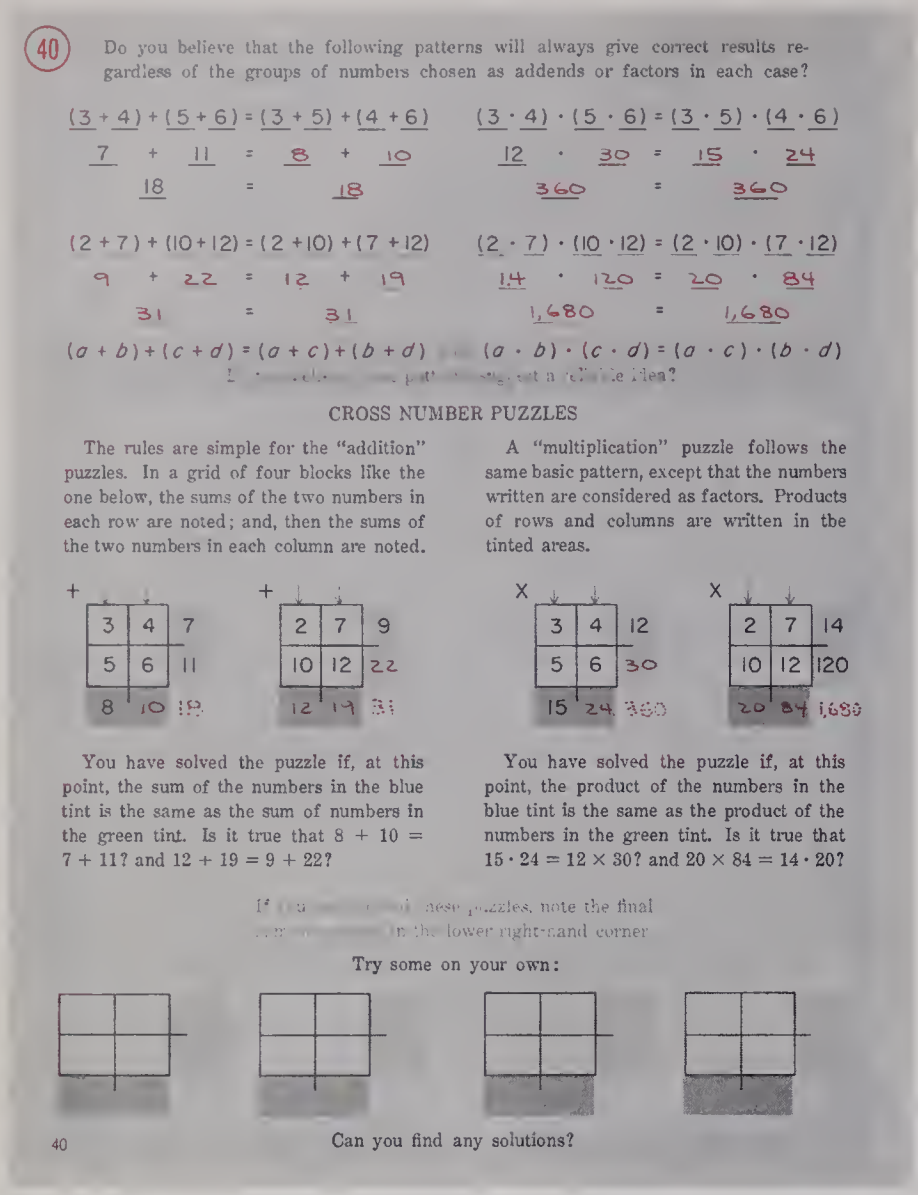
7 + 6 + 11 =
↓
13 + 11 = 24

In these examples, we grouped the two addends on the left. We repeat the examples, grouping the two on the right:

7 + 11 + 6 =
↓
7 + 17 = 24

7 + 6 + 11 =
↓
7 + 17 = 24

After approaching three different sets or groups of three addends, we are persuaded that order of



40 Do you believe that the following patterns will always give correct results regardless of the groups of numbers chosen as addends or factors in each case?

$(3+4)+(5+6)=(3+5)+(4+6)$ $\quad$ $(3\cdot4)\cdot(5\cdot6)=(3\cdot5)\cdot(4\cdot6)$

$7+11=8+10$ $\quad$ $12\cdot30=15\cdot24$

$18=18$ $\quad$ $360=360$

$(2+7)+(10+12)=(2+10)+(7+12)$ $\quad$ $(2\cdot7)\cdot(10\cdot12)=(2\cdot10)\cdot(7\cdot12)$

$9+22=12+19$ $\quad$ $14\cdot120=20\cdot84$

$31=31$ $\quad$ $1{,}680=1{,}680$

$(a+b)+(c+d)=(a+c)+(b+d)$ $\quad$ $(a\cdot b)\cdot(c\cdot d)=(a\cdot c)\cdot(b\cdot d)$

[illegible] pattern suggest a reliable idea?

CROSS NUMBER PUZZLES

The rules are simple for the "addition" puzzles. In a grid of four blocks like the one below, the sums of the two numbers in each row are noted; and, then the sums of the two numbers in each column are noted.

A "multiplication" puzzle follows the same basic pattern, except that the numbers written are considered as factors. Products of rows and columns are written in tbe tinted areas.

+			
	3	4	7
	5	6	11
	8	10	18

+			
	2	7	9
	10	12	22
	12	19	31

X			
	3	4	12
	5	6	30
	15	24	360

X			
	2	7	14
	10	12	120
	20	84	1,680

You have solved the puzzle if, at this point, the sum of the numbers in the blue tint is the same as the sum of numbers in the green tint. Is it true that $8 + 10 = 7 + 11$? and $12 + 19 = 9 + 22$?

You have solved the puzzle if, at this point, the product of the numbers in the blue tint is the same as the product of the numbers in the green tint. Is it true that $15 \cdot 24 = 12 \times 30$? and $20 \times 84 = 14 \cdot 20$?

If you [illegible] these puzzles, note the final [illegible] in the lower right-hand corner

Try some on your own:

Can you find any solutions?

40

Work Text page 39 develops part of the idea in parallel columns, to demonstrate that the pattern for addition has its counterpart for multiplication.

Note: In the expression $(a \cdot b)\ c = a\ (b \cdot c)$, point out that we often omit the multiplication sign when it would not lead to confusion:

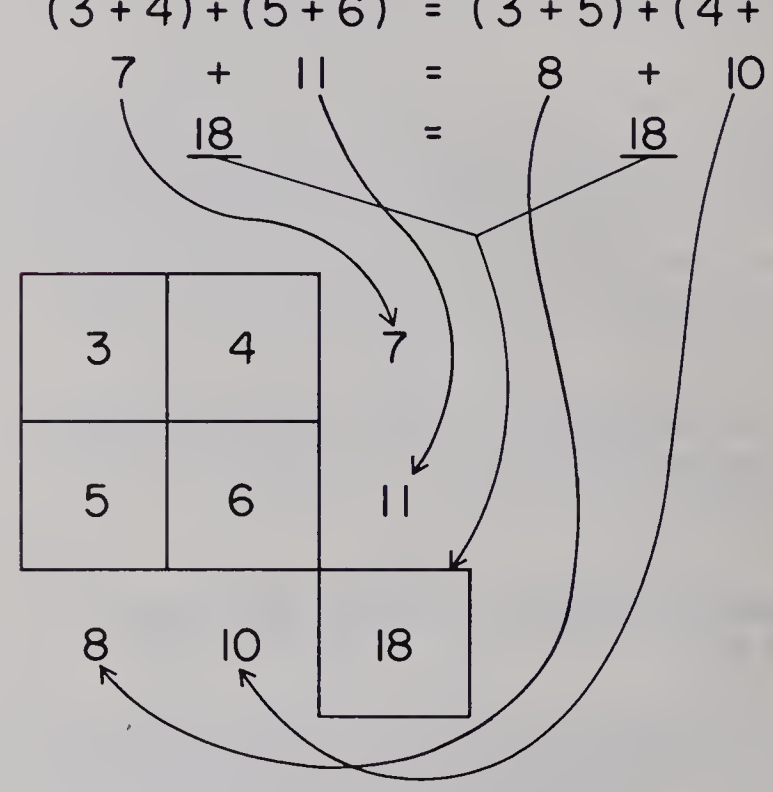
$$7 \times (3 \times 2) = 7\ (3 \times 2)$$
$$a \times (b \cdot c) = a\ (b \cdot c)$$

In fact, we can leave out the sign in the parentheses, and write:

$$a \times (b \times c) = a\ (bc)$$

and know that the multiplication is implied by agreement.

Work Text page 40 expands the pattern to examples in which sets of four addends and four factors are considered. The two examples of addition and the two examples of multiplication are repeated in Cross Number Puzzle form. Have children point out step-by-step the parallelism between the two forms:



and the same for the examples of multiplication.

listing the addends and an order of grouping the addends in pairs does not effect the outcome. The first part of this idea is quite obvious:

$$7 + 11 + 6 = 11 + 7 + 6, \text{ etc.}$$
$$9 + 12 + 15 = 15 + 9 + 12, \text{ etc.}$$

But, the other part of the "any order" idea is not as obvious to the uninitiated:

$$\underline{11 + 7} + 6 = 11 + \underline{7 + 6}$$
$$\downarrow \qquad\qquad \downarrow$$
$$\underline{18 \quad + 6} = \underline{11 + \ 13}$$
$$\downarrow \qquad \downarrow$$
$$24 = 24$$

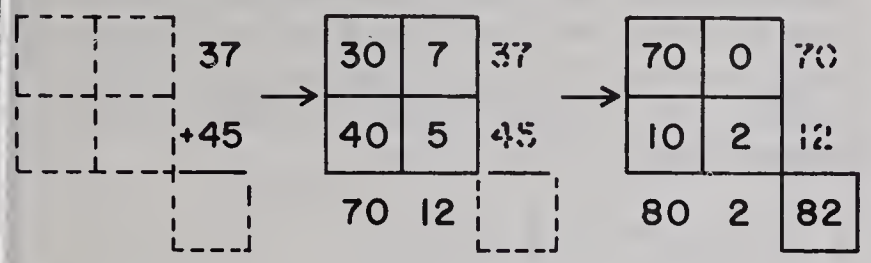
Passing through that second step—$18 + 6 = 11 + 13$—requires some analysis (or faith) that the two names that do not resemble each other actually are names for the same number. We are reassured by the next step in which the names come into focus as the same name for the same number.

(When the "in-any-order" rules are thinly disguised in the Cross Number Puzzle, they are far from obvious. Many adults are amazed when, in this form, they rediscover that a group of addends can be added in any order without influencing the sum.)

In the upper left hand corner of *Work Text* page 41, the pattern of the Cross Number Puzzles are, in a sense, named—indicated, at least. And for almost everyone, the mystery of the puzzles are solved. A few versions of the puzzle are indicated on the rest of page 41—which includes some always-needed drill in basic facts and operations.

If there is any mystery left in the addition form of the puzzle, it will be thoroughly dispelled by examples such as the one at the top of *Work Text* page 42. Here the abstract character of numbers is stripped away by using collections of objects. The activity outlined amounts simply to sweeping the same collections of objects together in two different ways—and, no one expects the total number of objects to change in the process.

When we carry through the usual algorism of addition, we go through these steps:

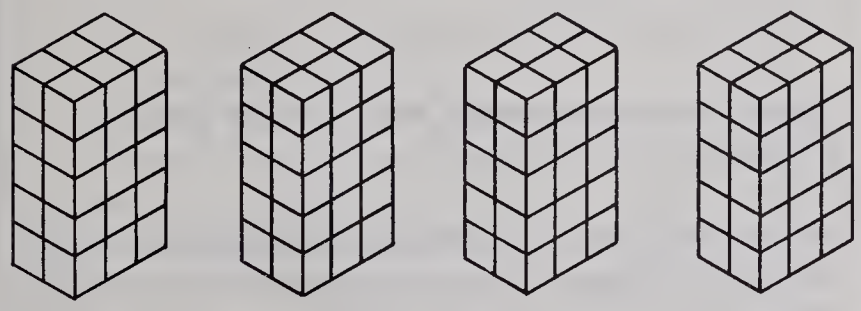


Work Text page 43 pictorially suggests the "associative principle for multiplication"—when there are three factors, the order in which they are paired for multiplication does not effect the final product:

$$3 \times 4 \times 5 = 3 \times 4 \times 5$$
$$12 \times 5 = 3 \times 20$$
$$(a \times b) \times c = a \times (b \times c)$$

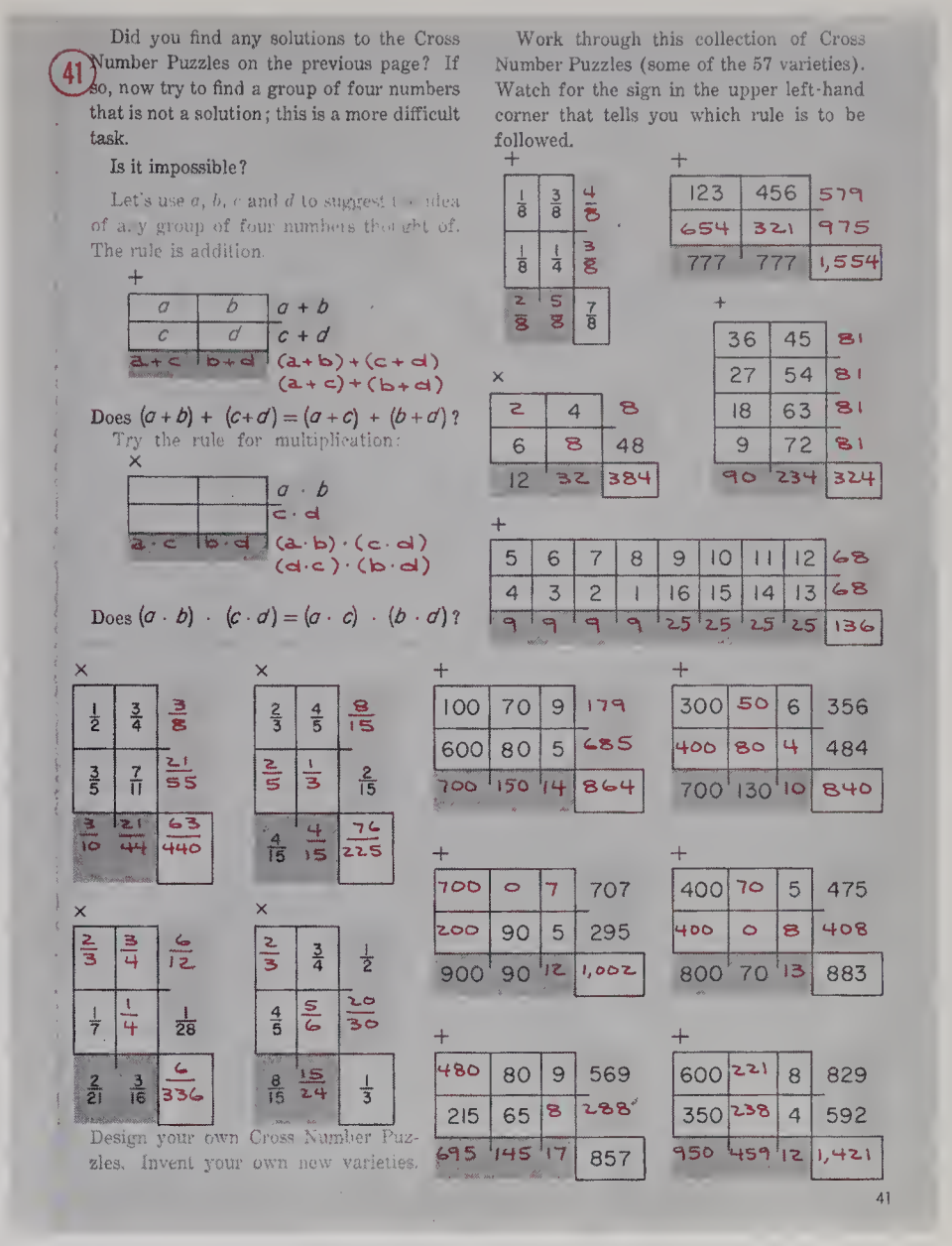
In computing volume, we consider each of the three dimensions as a factor, and volume as a product. We can express the idea as: Volume = length × width × height (V = lwh). And we soon learn that the order in which we make our computations does not effect the outcome.

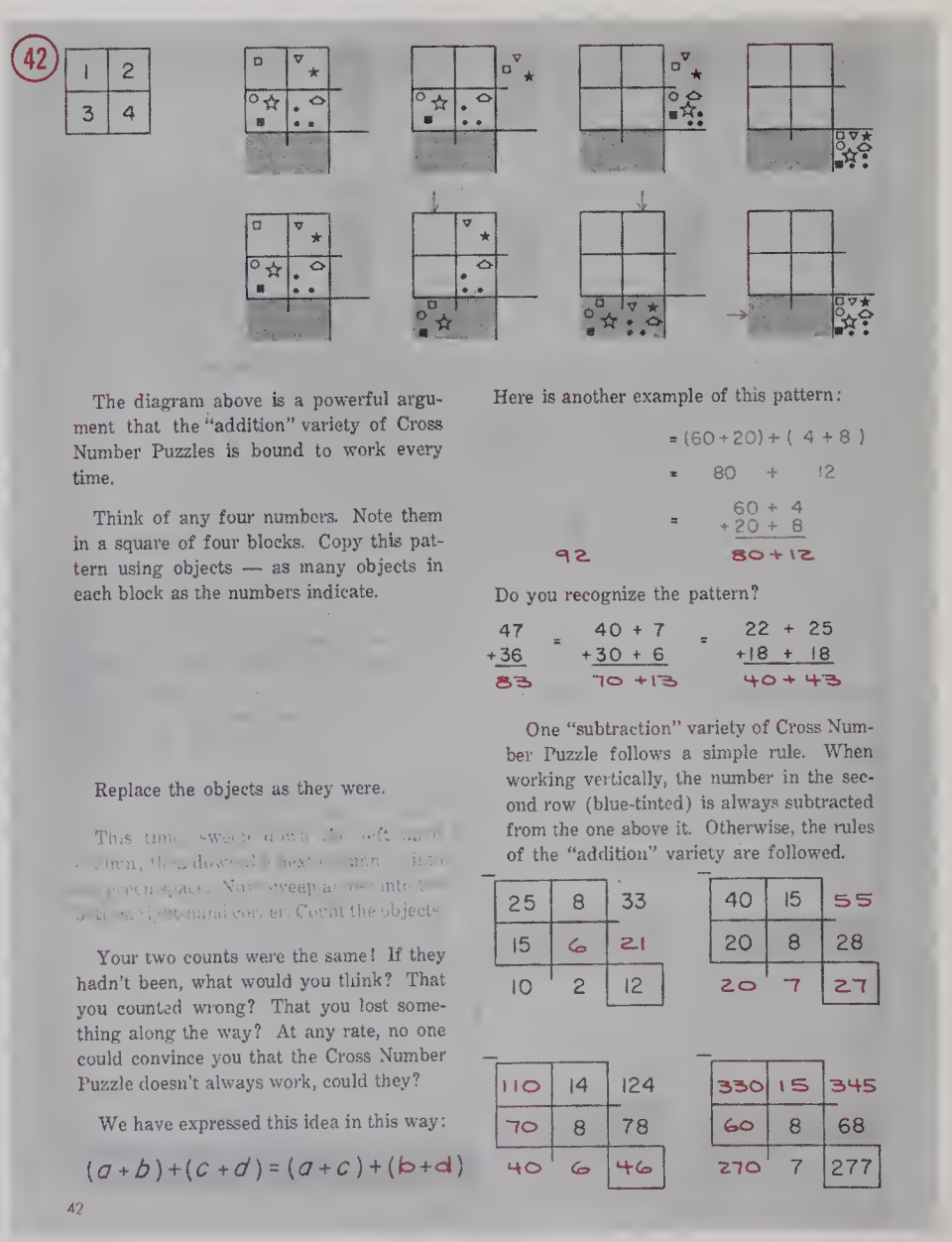
A pictorial representation of the idea involved when there are four factors can be an extension of the development on page 43:

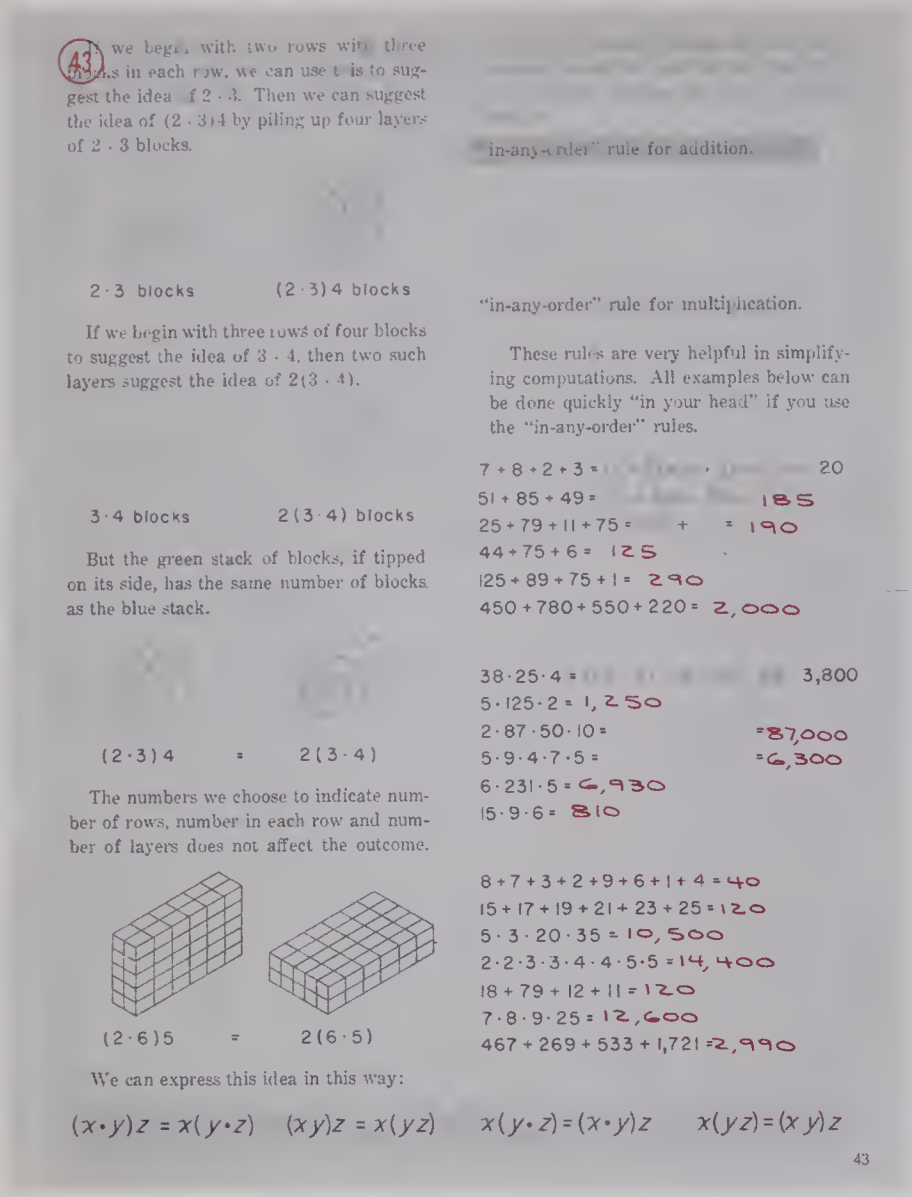


Each stack suggests the idea of 2 × 3 × 5. Four stacks suggests the idea of 4 × 2 × 3 × 5. In analyzing each situation, we find that order in multiplication is immaterial.

Sometimes, we do not realize that multiplication is a particular relationship between two numbers. Each of the basic facts we learn involves a pair of numbers that point to a third. When we learn the algorism for multiplication to help us find that


41
Did you find any solutions to the Cross Number Puzzles on the previous page? If so, now try to find a group of four numbers that is not a solution; this is a more difficult task.
Is it impossible?
Let's use a, b, c and d to suggest the idea of any group of four numbers thought of. The rule is addition.
Does (a + b) + (c + d) = (a + c) + (b + d)?
Try the rule for multiplication:
Does (a · b) · (c · d) = (a · c) · (b · d)?
Work through this collection of Cross Number Puzzles (some of the 57 varieties). Watch for the sign in the upper left-hand corner that tells you which rule is to be followed.
Design your own Cross Number Puzzles. Invent your own new varieties.
41


42
The diagram above is a powerful argument that the "addition" variety of Cross Number Puzzles is bound to work every time.
Think of any four numbers. Note them in a square of four blocks. Copy this pattern using objects — as many objects in each block as the numbers indicate.
Replace the objects as they were.
Your two counts were the same! If they hadn't been, what would you think? That you counted wrong? That you lost something along the way? At any rate, no one could convince you that the Cross Number Puzzle doesn't always work, could they?
We have expressed this idea in this way:
(a+b)+(c+d)=(a+c)+(b+d)
Here is another example of this pattern:
= (60+20)+(4+8)
= 80 + 12
= 60 + 4 + 20 + 8
92
80+12
Do you recognize the pattern?
47 + 36 = 40 + 7 + 30 + 6 = 22 + 25 + 18 + 18
83
70 + 13
40 + 43
One "subtraction" variety of Cross Number Puzzle follows a simple rule. When working vertically, the number in the second row (blue-tinted) is always subtracted from the one above it. Otherwise, the rules of the "addition" variety are followed.
42



43

we begin with two rows with three blocks in each row, we can use this to suggest the idea of 2 · 3. Then we can suggest the idea of (2 · 3)4 by piling up four layers of 2 · 3 blocks.

2 · 3 blocks (2 · 3)4 blocks

If we begin with three rows of four blocks to suggest the idea of 3 · 4, then two such layers suggest the idea of 2(3 · 4).

3 · 4 blocks 2(3 · 4) blocks

But the green stack of blocks, if tipped on its side, has the same number of blocks as the blue stack.

(2 · 3)4 = 2(3 · 4)

The numbers we choose to indicate number of rows, number in each row and number of layers does not affect the outcome.

(2 · 6)5 = 2(6 · 5)

We can express this idea in this way:

$(x \cdot y)z = x(y \cdot z)$ $(xy)z = x(yz)$ $x(y \cdot z) = (x \cdot y)z$ $x(yz) = (xy)z$

"in-any-order" rule for addition.

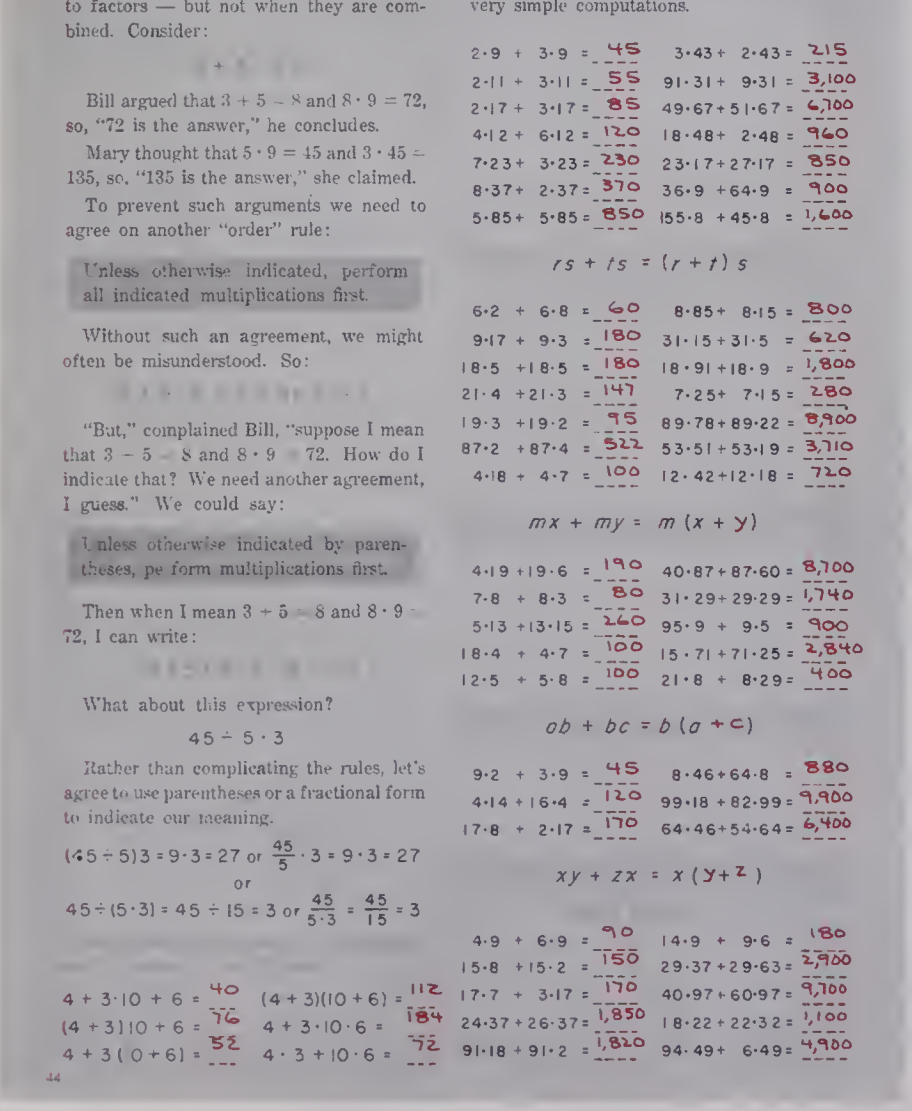
"in-any-order" rule for multiplication.

These rules are very helpful in simplifying computations. All examples below can be done quickly "in your head" if you use the "in-any-order" rules.

7 + 8 + 2 + 3 = 20
51 + 85 + 49 = 185
25 + 79 + 11 + 75 = + = 190
44 + 75 + 6 = 125
125 + 89 + 75 + 1 = 290
450 + 780 + 550 + 220 = 2,000

38 · 25 · 4 = 3,800
5 · 125 · 2 = 1,250
2 · 87 · 50 · 10 = =87,000
5 · 9 · 4 · 7 · 5 = =6,300
6 · 231 · 5 = 6,930
15 · 9 · 6 = 810

8 + 7 + 3 + 2 + 9 + 6 + 1 + 4 = 40
15 + 17 + 19 + 21 + 23 + 25 = 120
5 · 3 · 20 · 35 = 10,500
2 · 2 · 3 · 3 · 4 · 4 · 5 · 5 = 14,400
18 + 79 + 12 + 11 = 120
7 · 8 · 9 · 25 = 12,600
467 + 269 + 533 + 1,721 = 2,990



44

The "in-any-order" rules apply only when an expression is limited either to addends or to factors — but not when they are combined. Consider:

Bill argued that 3 + 5 = 8 and 8 · 9 = 72, so, "72 is the answer," he concludes.

Mary thought that 5 · 9 = 45 and 3 · 45 = 135, so, "135 is the answer," she claimed.

To prevent such arguments we need to agree on another "order" rule:

Unless otherwise indicated, perform all indicated multiplications first.

Without such an agreement, we might often be misunderstood. So:

"But," complained Bill, "suppose I mean that 3 + 5 = 8 and 8 · 9 = 72. How do I indicate that? We need another agreement, I guess." We could say:

Unless otherwise indicated by parentheses, perform multiplications first.

Then when I mean 3 + 5 = 8 and 8 · 9 = 72, I can write:

What about this expression?

45 ÷ 5 · 3

Rather than complicating the rules, let's agree to use parentheses or a fractional form to indicate our meaning.

(45 ÷ 5)3 = 9 · 3 = 27 or $\frac{45}{5} \cdot 3 = 9 \cdot 3 = 27$

or

45 ÷ (5 · 3) = 45 ÷ 15 = 3 or $\frac{45}{5 \cdot 3} = \frac{45}{15} = 3$

4 + 3 · 10 + 6 = 40 (4 + 3)(10 + 6) = 112
(4 + 3)10 + 6 = 76 4 + 3 · 10 · 6 = 184
4 + 3(10 + 6) = 52 4 · 3 + 10 · 6 = 72

If you discover the pattern in each of the groups below, the examples involve only very simple computations.

2 · 9 + 3 · 9 = 45 3 · 43 + 2 · 43 = 215
2 · 11 + 3 · 11 = 55 91 · 31 + 9 · 31 = 3,100
2 · 17 + 3 · 17 = 85 49 · 67 + 51 · 67 = 6,700
4 · 12 + 6 · 12 = 120 18 · 48 + 2 · 48 = 960
7 · 23 + 3 · 23 = 230 23 · 17 + 27 · 17 = 850
8 · 37 + 2 · 37 = 370 36 · 9 + 64 · 9 = 900
5 · 85 + 5 · 85 = 850 155 · 8 + 45 · 8 = 1,600

$rs + ts = (r + t)s$

6 · 2 + 6 · 8 = 60 8 · 85 + 8 · 15 = 800
9 · 17 + 9 · 3 = 180 31 · 15 + 31 · 5 = 620
18 · 5 + 18 · 5 = 180 18 · 91 + 18 · 9 = 1,800
21 · 4 + 21 · 3 = 147 7 · 25 + 7 · 15 = 280
19 · 3 + 19 · 2 = 95 89 · 78 + 89 · 22 = 8,900
87 · 2 + 87 · 4 = 522 53 · 51 + 53 · 19 = 3,710
4 · 18 + 4 · 7 = 100 12 · 42 + 12 · 18 = 720

$mx + my = m(x + y)$

4 · 19 + 19 · 6 = 190 40 · 87 + 87 · 60 = 8,700
7 · 8 + 8 · 3 = 80 31 · 29 + 29 · 29 = 1,740
5 · 13 + 13 · 15 = 260 95 · 9 + 9 · 5 = 900
18 · 4 + 4 · 7 = 100 15 · 71 + 71 · 25 = 2,840
12 · 5 + 5 · 8 = 100 21 · 8 + 8 · 29 = 400

$ab + bc = b(a + c)$

9 · 2 + 3 · 9 = 45 8 · 46 + 64 · 8 = 880
4 · 14 + 16 · 4 = 120 99 · 18 + 82 · 99 = 9,900
17 · 8 + 2 · 17 = 170 64 · 46 + 54 · 64 = 6,400

$xy + zx = x(y + z)$

4 · 9 + 6 · 9 = 90 14 · 9 + 9 · 6 = 180
15 · 8 + 15 · 2 = 150 29 · 37 + 29 · 63 = 2,900
17 · 7 + 3 · 17 = 170 40 · 97 + 60 · 97 = 9,700
24 · 37 + 26 · 37 = 1,850 18 · 22 + 22 · 32 = 1,100
91 · 18 + 91 · 2 = 1,820 94 · 49 + 6 · 49 = 4,900

$17 \times 23 = 391$, we are again working with two factors. When we are confronted with $3 \times 2 \times 5$, we see that there are three factors involved, and our concept of multiplication does not fit until we break down the expression and begin by selecting one pair out of the three factors to start with. If we select the first two written on the left of the expression, we indicate this with parentheses, and think:

$$(3 \times 2) \times 5 = 6 \times 5 = \_\_\_\_$$

And we have simplified the expression so we see a pair of factors: we know that $6 \times 5 = 30$. But we might have started by seeing the pair of factors written on the right side of the expression $3 \times 2 \times 5$, and we think:

$$3 \times (2 \times 5) = 3 \times 10 = \_\_\_\_$$

So, we have made the expression more manageable $-3 \times 10 = 30-$ because we are concerned with again only a pair of factors.

In both instances, the outcome was the same. We express that fact by saying:

$$(3 \times 2) \times 5 = 3 \times (2 \times 5)$$

and, in general, by writing:

$$(a \times b) \times c = a \times (b \times c)$$

The basic ideas of arithmetic become more apparent when we realize that in both basic operations — addition and multiplication — we must, by some means, look at only a pair of addends or a pair of factors at a time.

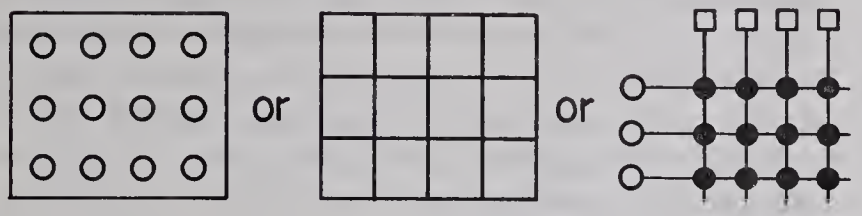
Notes to accompany *Work Text* pages 44-47

The next four pages delve more deeply into an idea met earlier many times, (on *Work Text* pages 18, 19 and 20 in particular). Ben Harris, in *Level D*, calls it his "Do-It-to-Both" rule. Mathematicians refer to it as the "distributive principle" — full name: "the distributive principle of multiplication over addition."

When we start out with the idea of addition, we learn about its patterns. Later we learn the patterns of multiplication. Each operation — addition and multiplication — can stand on its own feet.

The essential patterns of arithmetic are seen more clearly when multiplication is seen in its row-column representation rather than as "repetitive addition." The *Math Workshop* has favored the row-column concept for this reason. The idea of 3×4 as repetitive addition can be suggested this way:

(However, in life experiences, we more often meet the pattern of 3×4 in a row-column context which can be suggested by:



(When this approach to multiplication is emphasized, the idea of multiplication is developed without using the idea of addition; all it requires is a model such as one of the above and the idea of counting.)

But, these two independent operations which reflect patterns in life eventually meet—and are combined in statements such as:

$$3 \times (7+4) = 3 \times 7 + 3 \times 4$$
$$7 \times (20+6) = 7 \times 20 + 7 \times 6$$

or, in general:

$$a \times (b+c) = a \times b + a \times c, \text{ or}$$
$$a\,(b+c) = ab + ac, \text{ and}$$
$$ab + ac = a\,(b+c)$$

In this pattern, the ideas of addition and multiplication are interwoven.

Work Text page 44 explores this idea first as a shortcut. A clear cut example of this is illustrated by examples such as finding the result of $8 \cdot 7 + 8 \cdot 3$ which can be approached in two ways:

$$\text{(a) } 8 \cdot 7 + 8 \cdot 3 = 56 + 24 = 80$$
$$\text{(b) } 8 \cdot 7 + 8 \cdot 3 = 8(7+3) = 8 \cdot 10 = 80$$

Or the two approaches to $17 \cdot 42 + 17 \cdot 58$:

$$\text{(a) } 17 \cdot 42 + 17 \cdot 58 = 714 + 986 = 1700$$
$$\text{(b) } 17 \cdot 42 + 17 \cdot 58 = 17(42+58) = 17 \cdot 100 = 1700$$

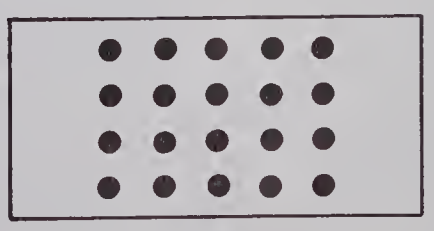
In fact, (b) above can be handled quickly with mental computations, the hardest of which is $42 + 58 = 100$.

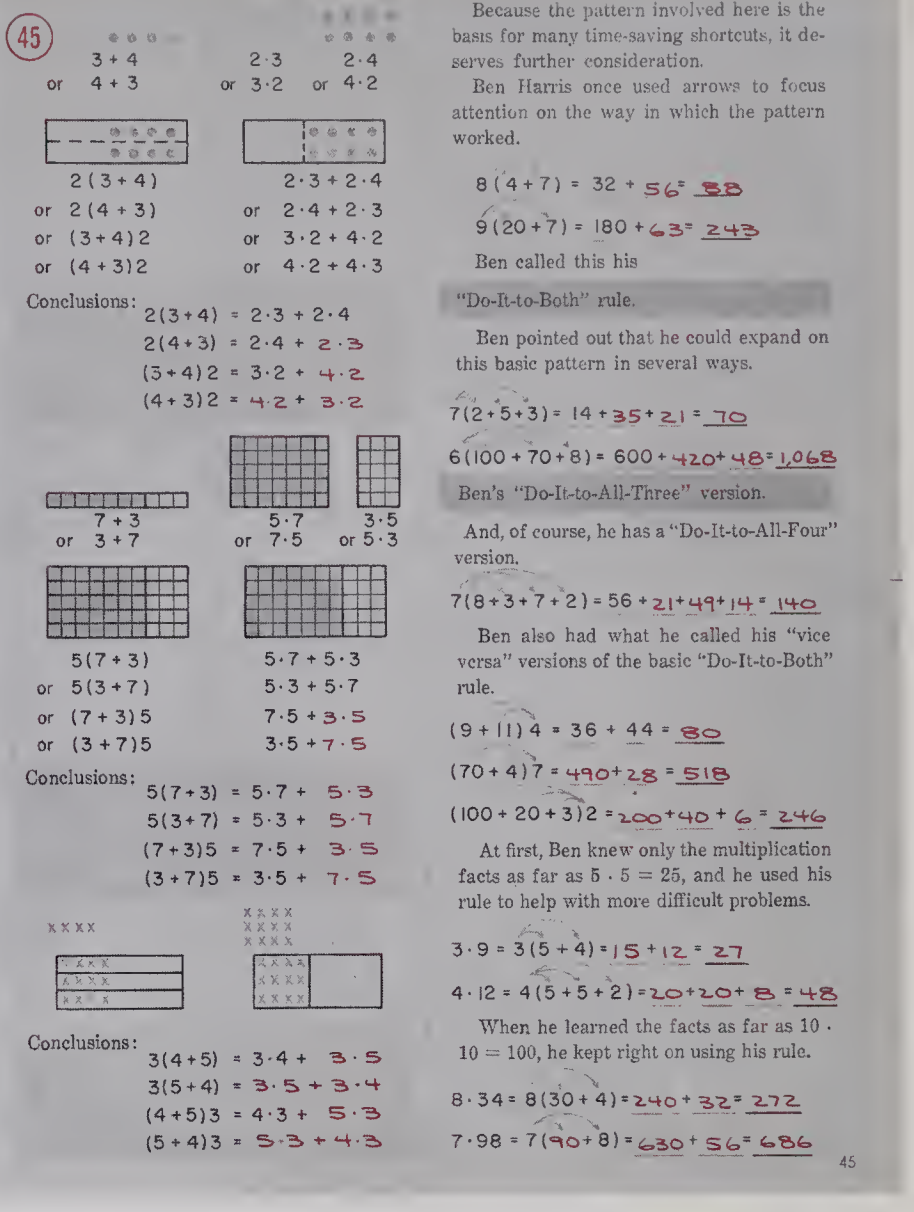
When the idea of the shortcut has been introduced on *Work Text* page 44, ask the children to make up examples that will help to emphasize the shortcut offered in this pattern. They may offer examples such as:

$$395.17 \cdot 492 + 395.17 \cdot 508 = 395{,}170, \text{ and}$$
$$1{,}428 \cdot 286 + 572 \cdot 286 = 2{,}000 \cdot 286 = 572{,}000$$

As the children vie with one another to find complex situations to demonstrate the usefulness of the idea, their understanding grows—and they invite cumbersome fractions in the use of large numbers.

Work Text page 45 makes a pictorial analysis of the "Do-It-to-Both" or "distributive" principle. Simple manipulative devices can be made to develop the concept. Here is one such device—simply an array of dots on a sheet of paper:




45
3 + 4
or 4 + 3
2 · 3
or 3 · 2
2 · 4
or 4 · 2
2(3 + 4)
or 2(4 + 3)
or (3 + 4)2
or (4 + 3)2
2 · 3 + 2 · 4
or 2 · 4 + 2 · 3
or 3 · 2 + 4 · 2
or 4 · 2 + 4 · 3
Conclusions:
2(3+4) = 2·3 + 2·4
2(4+3) = 2·4 + 2·3
(3+4)2 = 3·2 + 4·2
(4+3)2 = 4·2 + 3·2
7 + 3
or 3 + 7
5 · 7
or 7 · 5
3 · 5
or 5 · 3
5(7 + 3)
or 5(3 + 7)
or (7 + 3)5
or (3 + 7)5
5 · 7 + 5 · 3
5 · 3 + 5 · 7
7 · 5 + 3 · 5
3 · 5 + 7 · 5
Conclusions:
5(7+3) = 5·7 + 5·3
5(3+7) = 5·3 + 5·7
(7+3)5 = 7·5 + 3·5
(3+7)5 = 3·5 + 7·5
Conclusions:
3(4+5) = 3·4 + 3·5
3(5+4) = 3·5 + 3·4
(4+5)3 = 4·3 + 5·3
(5+4)3 = 5·3 + 4·3
Because the pattern involved here is the basis for many time-saving shortcuts, it deserves further consideration.
Ben Harris once used arrows to focus attention on the way in which the pattern worked.
8(4+7) = 32 + 56 = 88
9(20+7) = 180 + 63 = 243
Ben called this his
"Do-It-to-Both" rule.
Ben pointed out that he could expand on this basic pattern in several ways.
7(2+5+3) = 14 + 35 + 21 = 70
6(100+70+8) = 600 + 420 + 48 = 1,068
Ben's "Do-It-to-All-Three" version.
And, of course, he has a "Do-It-to-All-Four" version.
7(8+3+7+2) = 56 + 21 + 49 + 14 = 140
Ben also had what he called his "vice versa" versions of the basic "Do-It-to-Both" rule.
(9+11)4 = 36 + 44 = 80
(70+4)7 = 490 + 28 = 518
(100+20+3)2 = 200 + 40 + 6 = 246
At first, Ben knew only the multiplication facts as far as 5 · 5 = 25, and he used his rule to help with more difficult problems.
3·9 = 3(5+4) = 15 + 12 = 27
4·12 = 4(5+5+2) = 20 + 20 + 8 = 48
When he learned the facts as far as 10 · 10 = 100, he kept right on using his rule.
8·34 = 8(30+4) = 240 + 32 = 272
7·98 = 7(90+8) = 630 + 56 = 686
45

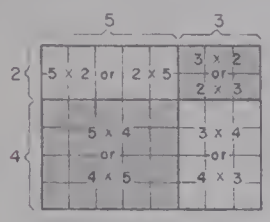
46 Another version of Ben's "Do-It-to-Both" rule he called "Double-Do-It-to-Both."

(5+3)(2+4) = 10 + 20 + 6 + 12 = 48

(8+5)(3+7) = 24 + 56 + 15 + 35 = 130

(10+4)(10+8) = 100 + 80 + 40 + 32 = 252

This expansion of the "Do-It-to-Both" rule can be suggested in diagrams such as the following:



There are two ways in which we can account for all the blocks in the diagram:

A – 8 6 48

B – + + + =

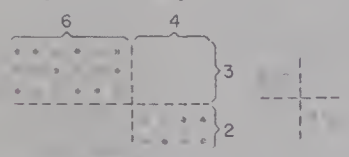
10 + 20 + 6 + 12 = 48

Conclusion:

(5+3)(2+4) = 5·2 + 5·4 + 3·2 + 3·4

This is the same conclusion Ben reached by his "Double-Do-It-to-Both" version.

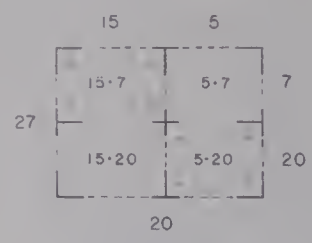
Let's try another example:



Conclusion:

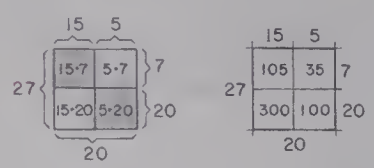
(6+4)(3+2) = 6·3 + 6·2 + 4·3 + 4·2

Examples can involve such large numbers that diagrams would take up too much space if drawn in detail. Here, then, is a summary of another example:

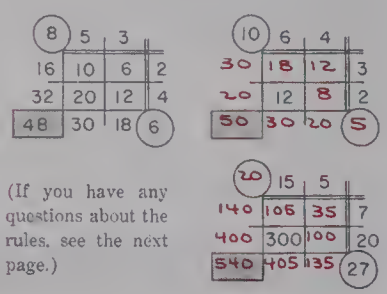


Broken green and blue lines above indicate that, if all the dots were drawn, there would be as many dots in that row or column as the number indicates.

An even more condensed diagram might be:

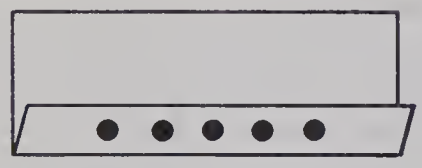


Ben, at this point, said that this suggested a version of the Cross Number Puzzle to keep a record of diagrams for his "Double-Do-It-to-Both" rule. He would record the three examples on the page in this way:

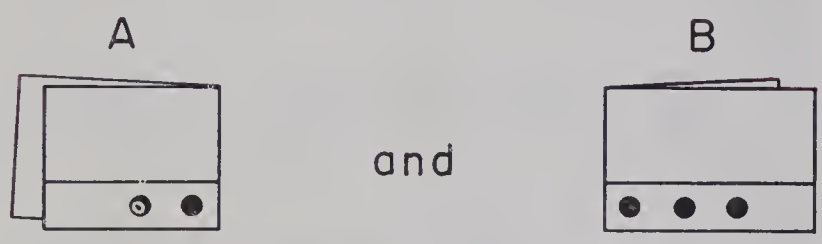


(If you have any questions about the rules, see the next page.)

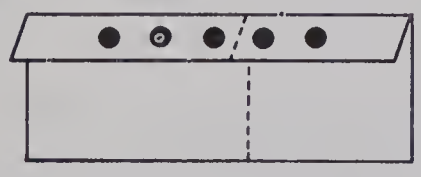
Fold it, showing only one row:



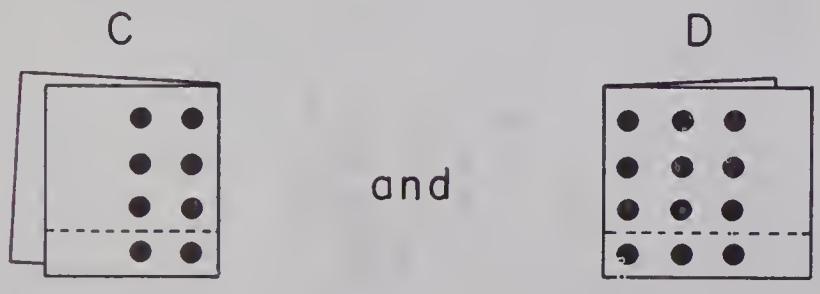
Fold again, so that the two sides look like this:



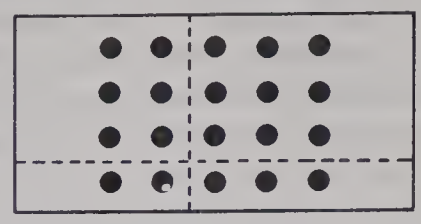
Show side A: "How many?" . . . 2. Show side B: "How many?" . . . 3. Open up:



"How many?" . . . 2 + 3 = 5. Open up, and refold as:

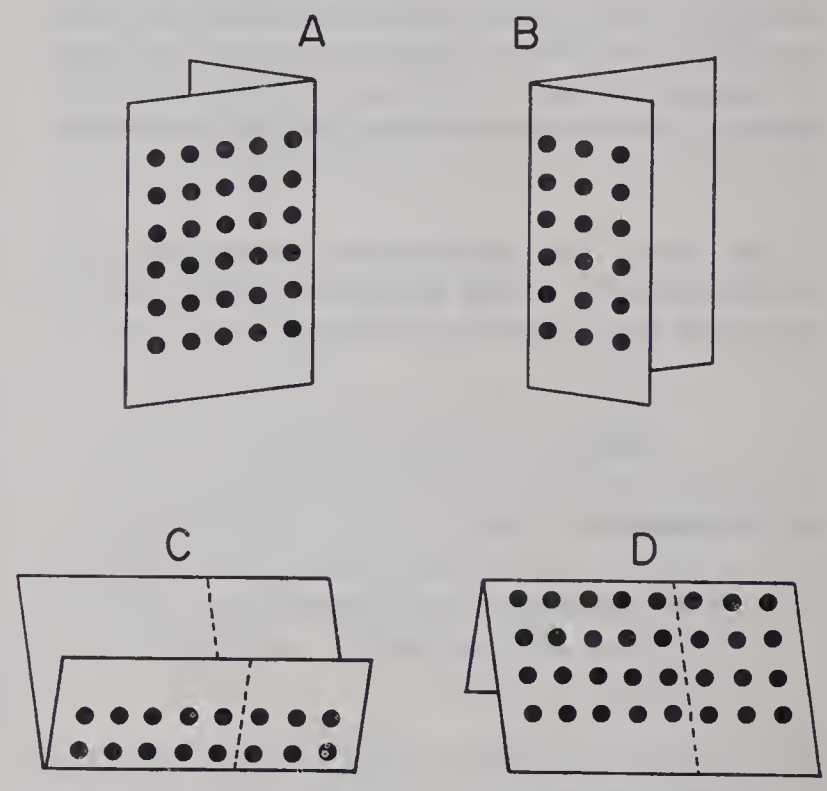


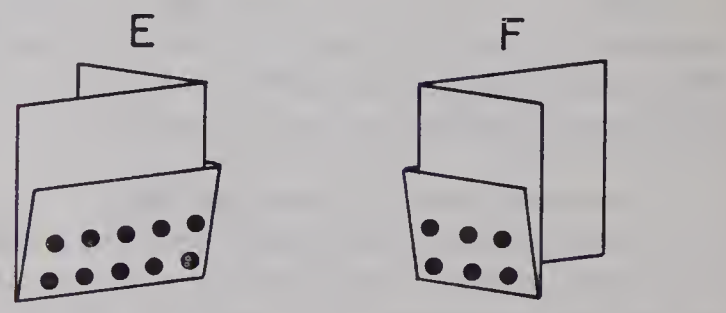
"How many on C?" . . . 4 × 2 = 8 "How many on D?" . . . 4 × 3 = 12 Open up:



"How many?" . . . 4 × 5 or 4 × (2+3) or 4 × 2 + 4 × 3 or 3 × 4 + 2 × 4 or 5 + 15 or 8 + 12, etc. Thus, pictorially, this pattern is suggested by such simple devices: 4 × 2 + 4 × 3 = 4 × (2+3)

A collection of such arrays or rectangular arrangements of dots or other marks, can be made in a short period of time to illustrate many examples of the idea that: a (b+c) = ab + ac. These same arrays can be used to suggest the expansion of this idea on *Work Text* pages 46 and 47. An array of 6 by 8 dots might be folded to illustrate the situation discussed on page 46:





A → 6×5 and B → 6×3 and A&B → 6·5 + 6·3 or 6(5+3)

C → 2×8 and D → 4×8 and C&D → 2·8 + 4·8 = (2+4)8

E → 2×5 and F → 2×3 and E&F → 2·5 and 2·3 = 2(5+3)

In similar ways, each of the sections of the array can be shown separately; in pairs of sections and as the whole array. The whole array can be used to suggest the idea of

$(2+4)\cdot(5+3)$ or $(2+4)\cdot 5 + (2+4)\,3$ or $2\,(5+3) + 4(5+3)$, and $2\cdot 5 + 2\cdot 3 + 4\cdot 5 + 4\cdot 3$
etc.

In the second column of *Work Text* page 46, a sketch is introduced "to save time drawing dots." This is an important step toward the sketch used at the top of the second column of page 47—a sketch that can be used to suggest the idea behind a whole series of examples. A Cross Number Puzzle form is used as a kind of scoreboard to keep a record of the number of dots or squares in each part of array that has been sectioned. Together, these devices help reveal the mechanics of the long multiplication in which we "regroup" each factor, and then weave a pattern of multiplications and additions to find the product. Similarly, they help develop an understanding of what goes on in the division algorism in which a similar pattern is woven of multiplications and additions.

Both important algorisms are based on the simple and fundamental pattern:

$$a\,(b+c) = ab + ac$$

and upon an extension of the idea suggested by the pattern indicator:

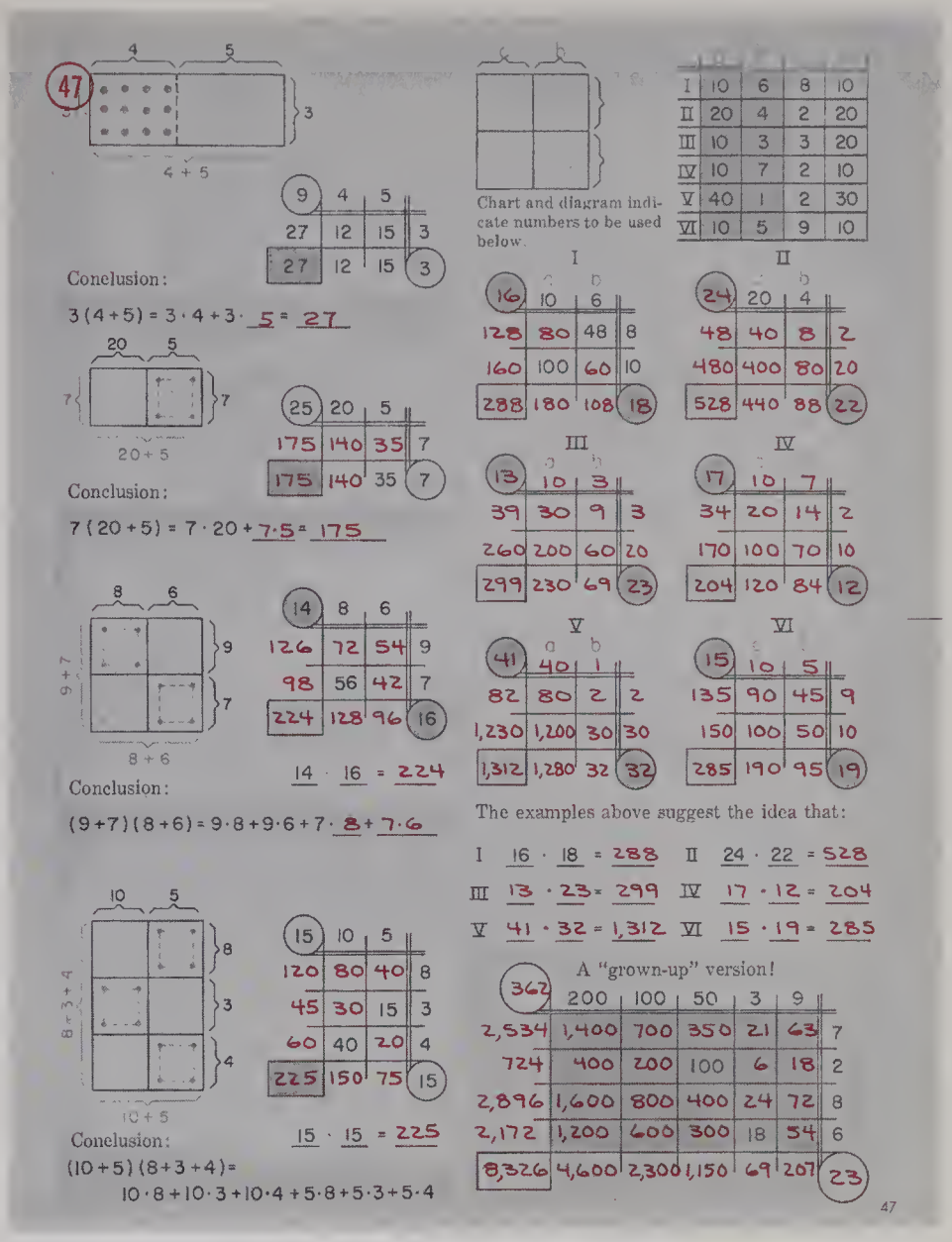
$$(b+c) \div a = b \div a + c \div a$$

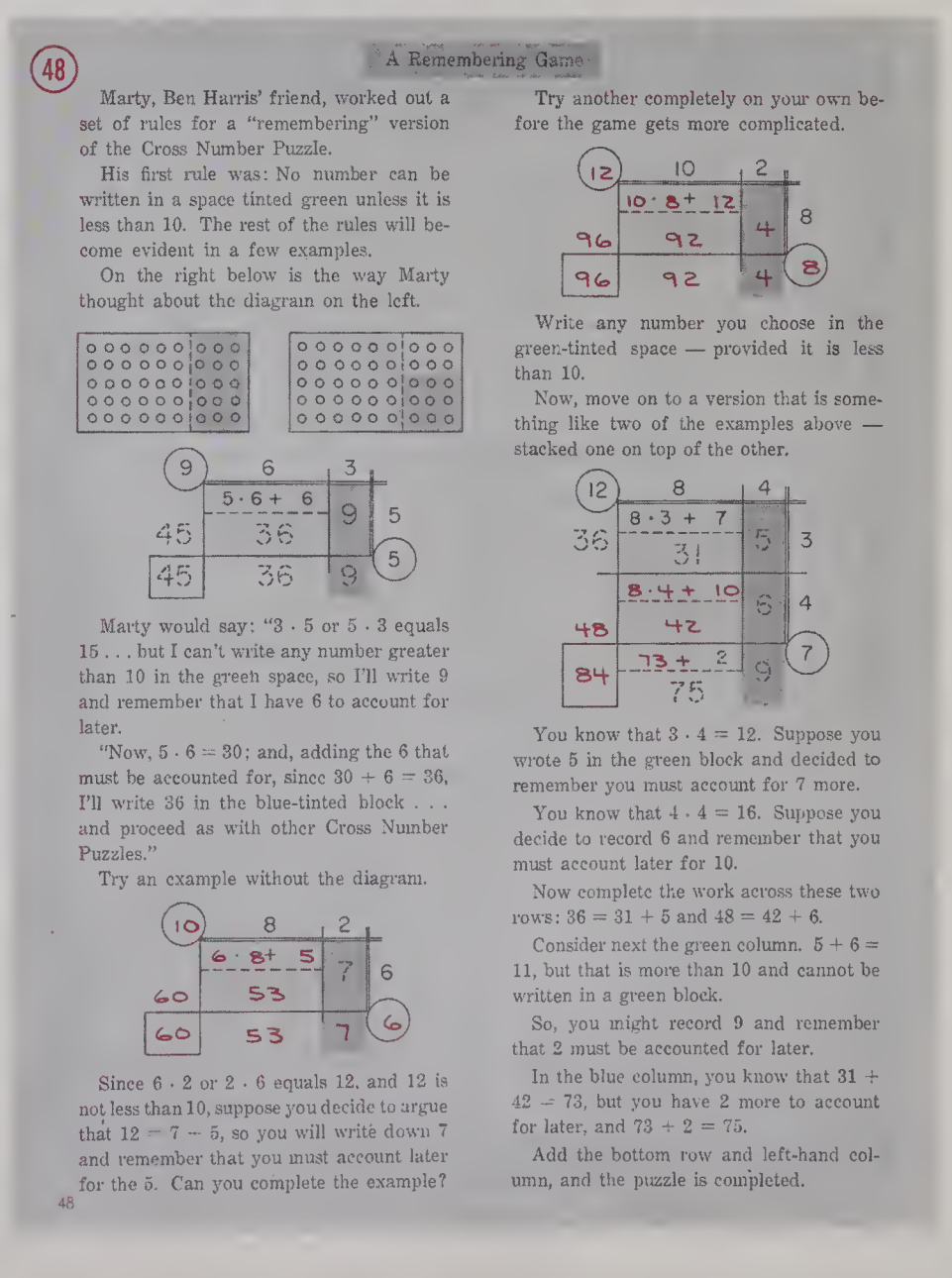
These ideas will be probed further and put to use in the pages ahead. Much space is devoted to the ramifications of the "distributive principle," because its role has not been considered by traditional textbooks or pedagogy. The failure to understand this principle lies at the roots of much confusion in children's minds. This confusion later becomes one of the great barriers to the understanding of algebra.

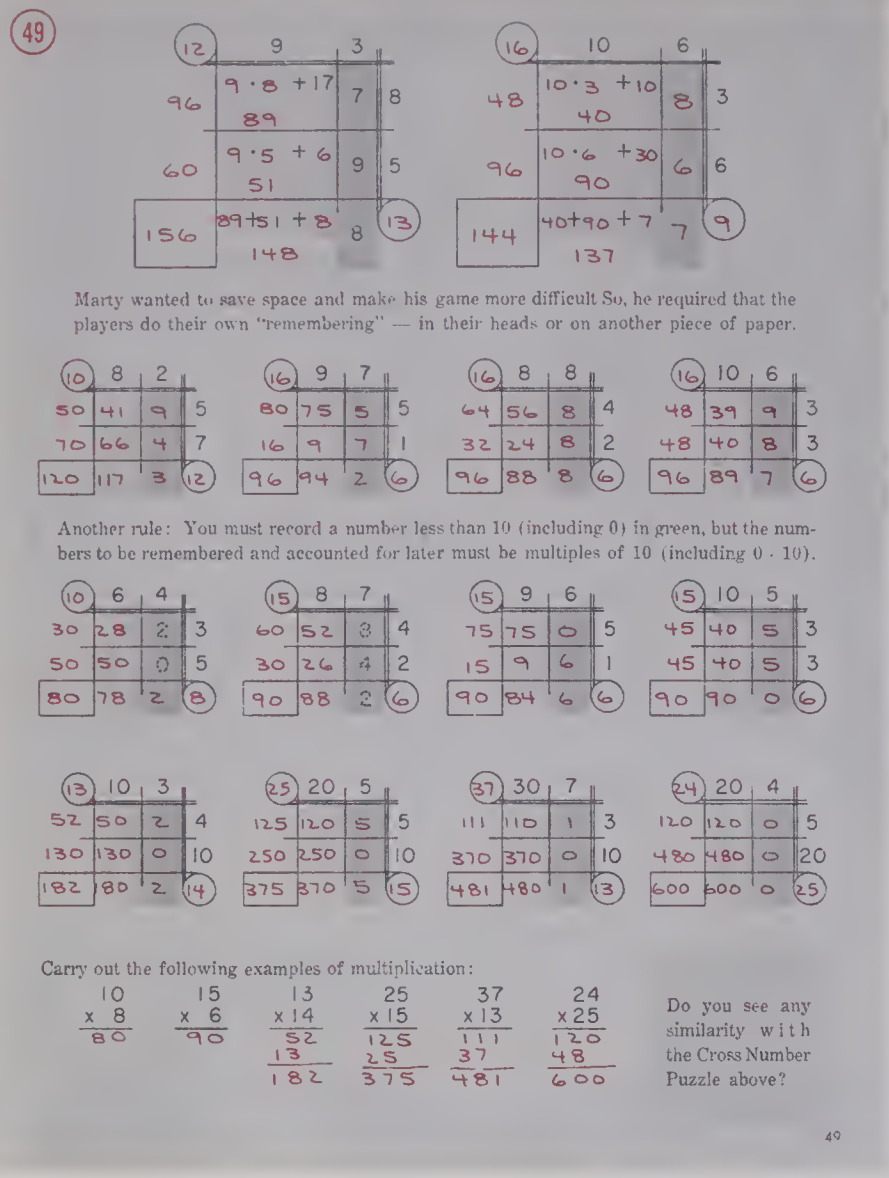
Notes to accompany *Work Text* pages 48-51

Analyze your thinking about examples of simple multiplication, such as 7×16. It is usually written:

$$\begin{array}{r} 16 \\ \times\ 7 \\ \hline \end{array}$$


47
Conclusion:
3(4+5) = 3·4+3·5 = 27
Conclusion:
7(20+5) = 7·20+7·5 = 175
Conclusion:
(9+7)(8+6) = 9·8+9·6+7·8+7·6
14 · 16 = 224
Conclusion:
(10+5)(8+3+4) =
10·8+10·3+10·4+5·8+5·3+5·4
15 · 15 = 225
Chart and diagram indicate numbers to be used below.
The examples above suggest the idea that:
I 16 · 18 = 288 II 24 · 22 = 528
III 13 · 23 = 299 IV 17 · 12 = 204
V 41 · 32 = 1,312 VI 15 · 19 = 285
A "grown-up" version!
47


48
A Remembering Game
Marty, Ben Harris' friend, worked out a set of rules for a "remembering" version of the Cross Number Puzzle.
His first rule was: No number can be written in a space tinted green unless it is less than 10. The rest of the rules will become evident in a few examples.
On the right below is the way Marty thought about the diagram on the left.
Marty would say: "3 · 5 or 5 · 3 equals 15 . . . but I can't write any number greater than 10 in the green space, so I'll write 9 and remember that I have 6 to account for later.
"Now, 5 · 6 = 30; and, adding the 6 that must be accounted for, since 30 + 6 = 36, I'll write 36 in the blue-tinted block . . . and proceed as with other Cross Number Puzzles."
Try an example without the diagram.
Since 6 · 2 or 2 · 6 equals 12, and 12 is not less than 10, suppose you decide to argue that 12 = 7 + 5, so you will write down 7 and remember that you must account later for the 5. Can you complete the example?
Try another completely on your own before the game gets more complicated.
Write any number you choose in the green-tinted space — provided it is less than 10.
Now, move on to a version that is something like two of the examples above — stacked one on top of the other.
You know that 3 · 4 = 12. Suppose you wrote 5 in the green block and decided to remember you must account for 7 more.
You know that 4 · 4 = 16. Suppose you decide to record 6 and remember that you must account later for 10.
Now complete the work across these two rows: 36 = 31 + 5 and 48 = 42 + 6.
Consider next the green column. 5 + 6 = 11, but that is more than 10 and cannot be written in a green block.
So, you might record 9 and remember that 2 must be accounted for later.
In the blue column, you know that 31 + 42 = 73, but you have 2 more to account for later, and 73 + 2 = 75.
Add the bottom row and left-hand column, and the puzzle is completed.
48


49
Marty wanted to save space and make his game more difficult So, he required that the players do their own "remembering" — in their heads or on another piece of paper.
Another rule: You must record a number less than 10 (including 0) in green, but the numbers to be remembered and accounted for later must be multiples of 10 (including 0 · 10).
Carry out the following examples of multiplication:
10 x 8 = 80
15 x 6 = 90
13 x 14 = 52, 13, 182
25 x 15 = 125, 25, 375
37 x 13 = 111, 37, 481
24 x 25 = 120, 48, 600
Do you see any similarity with the Cross Number Puzzle above?
49

When we start out saying to ourselves, "7 × 6 = 42," we have assumed a "regrouping" that has changed the problem to look like this:

$$\begin{array}{r} 10 + 6 \\ \times \quad 7 \\ \hline \end{array}$$

and this is equivalent to saying:

$$7 \times 16 = 7(10+6)$$

We say to ourselves, "7 times 6 = 42; put down the 2 and 'carry' or 'remember' 40," so the example then looks like this:

$$\begin{array}{r} 16 \\ \times \quad 7 \\ \hline \end{array}$$

2 (and remember 40)

And a more complete picture of what we have done is:

$$\begin{array}{r} 10 + 6 \\ \times \quad 7 \\ \hline (40+)2 \end{array} \rightarrow \begin{array}{r} 6 \\ \times 7 \\ \hline \end{array} \rightarrow 42 = 40 + 2 \quad \leftarrow$$

Then we say: "7 times 10 = 70; and 70 plus the 40 I remember or carry is 110; I'll write 11 next to 2 and get 112—which is the sum of 110 and 2."

$$\begin{array}{r} 16 \\ \times \quad 7 \\ \hline 116 \end{array}$$

If we were thinking our way through this process, each step could be indicated by successive expressions, and they might be:

$$\begin{aligned} 7 \times 16 &= 7(10 + 6) \\ &= 7 \times 10 + 7 \times 6 \\ &= 7 \times 10 + 42 \\ &= 7 \times 10 + 40 + 2 \\ &= 7 \times 10 + 4 \times 10 + 2 \\ &= (7+4) \times 10 + 2 \\ &= 11 \times 10 + 2 \\ &= 110 + 2 \\ &= 112 \end{aligned}$$

The 2 is italicized to indicate the point at which we wrote it down. In the last line, we write 11 in the "places" next to the 2—and produce 112 as the product. Now, let's go through that same process with a different regrouping of the factor 16. Instead of regrouping 16 as 10 + 6, we might regroup 16 as 7 + 9. In vertical form, we would argue:

$$\begin{array}{r} 16 \\ \times \quad 7 \\ \hline \end{array} \qquad \begin{array}{r} 7 + 9 \\ \times 7 \\ \hline \end{array}$$

60 + 3 (7 × 9 = 63 or 60 + 3; put down 3, carry 60)

40 + 9 (7 × 7 = 49 or 40 + 9; put down 9, carry 40—along with 60)

11 2 (3 + 9 = 12 or 10 + 2; put down 2 and remember 10—along with 60 and 40—or a total of 110. Since 110 and 2 is 112, all I need to do is write the 11 in the places to the left of 2.

Now, this form could be contrasted to look like this:

$$\begin{array}{r} 16 \\ \times \quad 7 \\ \hline \end{array} \qquad \begin{array}{r} 7 + 9 \\ \times 7 \\ \hline 63 \\ 49 \\ \hline 112 \end{array}$$

and the argument, step by step, might be:

$$\begin{aligned} 7 \times 16 &= 7(7+9) \\ &= 7 \times 7 + 7 \times 9 \\ &= 7 \times 7 + 63 \\ &= 7 \times 7 + 60 + 3 \\ &= 49 + 60 + 3 \\ &= 40 + 9 + 60 + 3 \\ &= 40 + 60 + 9 + 3 \\ &= 40 + 60 + 12 \\ &= 40 + 60 + 10 + 2 \\ &= 100 + 10 + 2 \\ &= 110 + 2 \\ &= 112 \end{aligned}$$

When confronted with 7×16, why do we regroup this as $7(10+6)$ rather than $7(9+7)$ or $7(11+5)$ or $7(8+8)$? Is one regrouping more "correct" than others? Clearly, that is not the case. All will bring out the same conclusion.

$10 + 6$	$9 + 7$	$11 + 5$	$8 + 8$	$12 + 4$
$\times 7$	$\times 7$	$\times 7$	$\times 7$	$\times 7$
42	49	35	56	28
70	63	77	56	84
112	112	112	112	112

After children understand that many regroupings are possible, then—and only then—are we prepared to answer the question: Why do we usually select the first regrouping shown?

It is a sheer convenience! By remembering, we can shorten the record keeping to:

$$\begin{array}{r} 16 \\ \times\ 7 \\ \hline 112 \end{array}$$

and no other example lends itself to any such convenience with so little remembering.

The "remembering games" on *Work Text* pages 48 and 49 become a little involved—and purposely so. Let the children struggle with them. Through a somewhat weird "remembering" system or systems, it is possible to reach sensible conclusions.

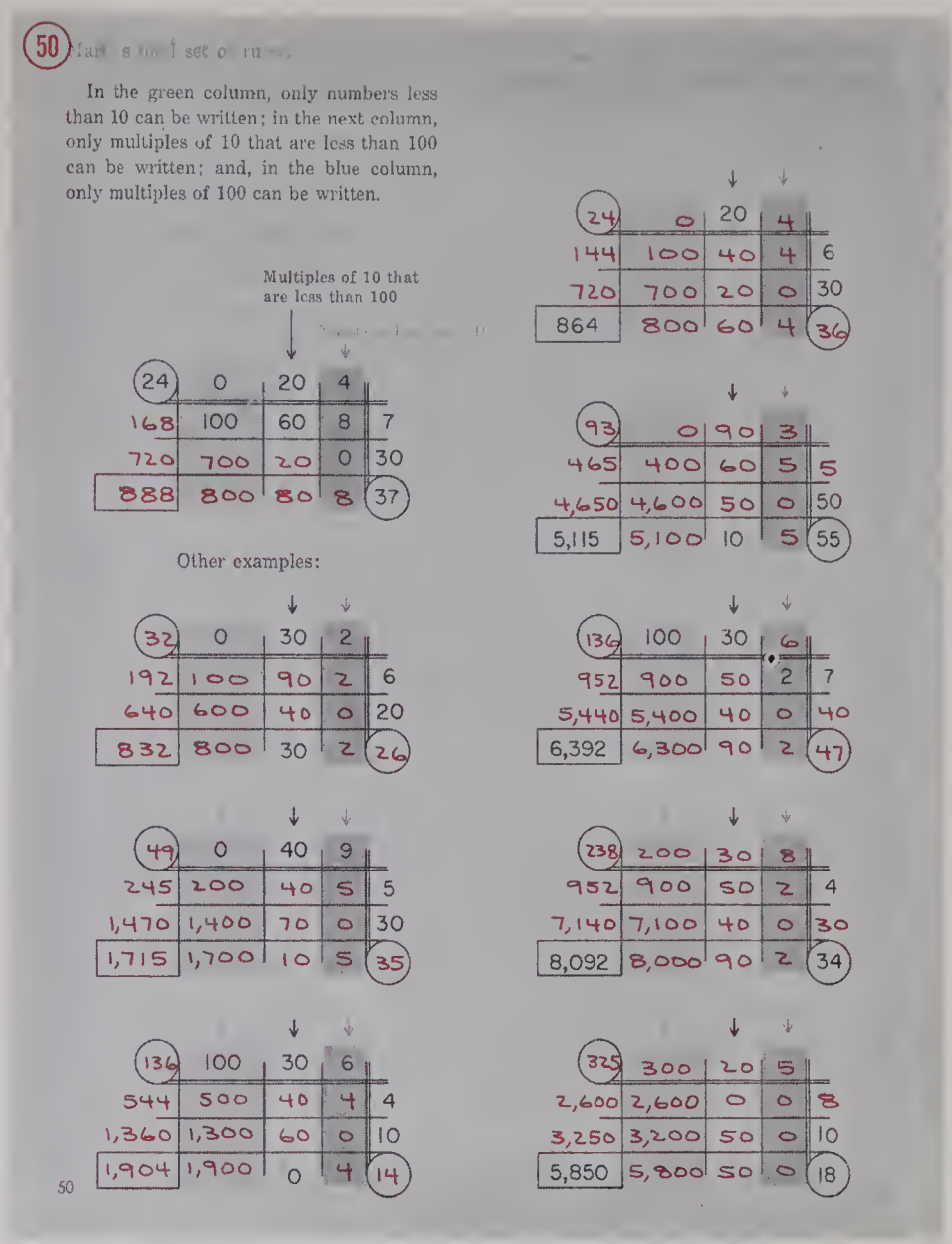
The game becomes a little easier as *Work Text* page 49 develops, and the bottom row of examples is the easiest of all. Why? Because in the regrouping of the factors, we took advantage of the idea of place value. Even the additions become much less complicated—more zeros are encountered.

By the time the last row is reached, the procedure is an almost perfect parallel with the traditional algorism of multiplication—and children understand more fully what is going on. The game now expands on the basic pattern of the bottom row of page 49. It now helps reduce such examples as 14×136 to manageable size.

Marty's game

A twist develops in the second column of examples of Marty's game. A pattern of "constructing" a Cross Number Puzzle becomes a problem of reconstruction. The number in the check box is given and enough information is given to see the regrouping of the initiating number on top or right side.

To reconstruct an example of multiplication from a given product and one factor—which is exactly


50
In the green column, only numbers less than 10 can be written; in the next column, only multiples of 10 that are less than 100 can be written; and, in the blue column, only multiples of 100 can be written.
Multiples of 10 that are less than 100
Other examples:
50
51
Complete the following:
Marty's game

the case at hand—is simply another name for "division."

So, the final example can be stated: 5,850 ÷ 18 = _____. Each step, if analyzed, has its counterpart in the traditional algorism of long division. To fill in the missing blue-tinted space on the top of the form, one must find a number to try by the same exact method of finding a "trial divisor"—but it is carried on with far more understanding.

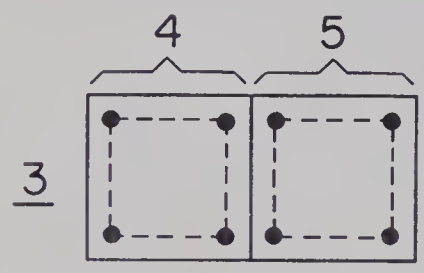
Work Text page 51 brings the games back to their simplest form, in the pattern of

$$a(b+c) = ab + ac, \text{ or}$$
$$a(b+c+d) = ab + ac + ad$$

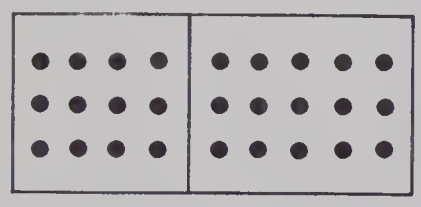
The emphasis in these games on page 51 is on the wide freedom there is in their pattern for a great variety of regroupings—and the games call for a selection from among the many, at least one of the variety that meeds the conditions of the game. The next four pages consider this same basic idea with more emphasis on names for the patterns and examples of the pattern from the world of events.

Notes to accompany *Work Text* pages 52-55

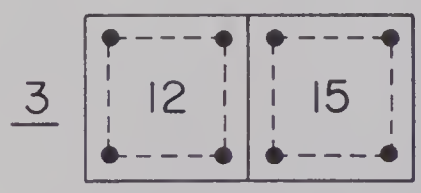
By now, the children will probably understand fully the meaning behind this sketch:



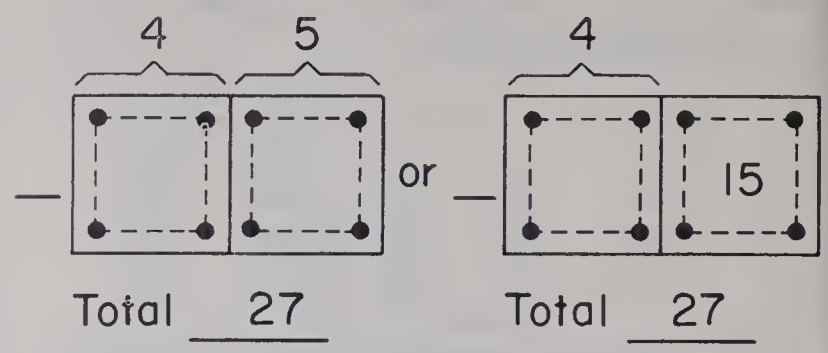
It is a kind of sketch shorthand for:



The same sketch is implied by:

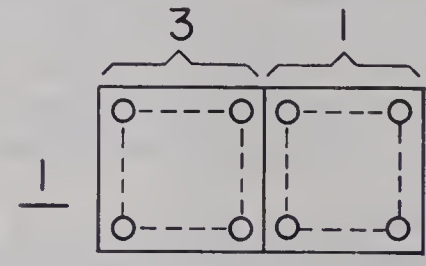


though we now need to know that if 12 objects are arranged in three rows, there are 4 in each row; and if 15 objects are arranged in three rows, there are 5 in each row. Another way of implying the same idea by a sketch is:



In one case, one needs to know that 4 + 5 = 9 and if 27 objects are arranged in rows of 9, there must be 3 rows. In the second, one needs to know that if 27 objects are arranged in two arrays and there are 15 in one array, there must be 12 in the other. Then, if those 12 are arranged so there are 4 in each row, then there must be 3 rows—and that 15 arranged in 3 rows indicates there are 5 in each row.

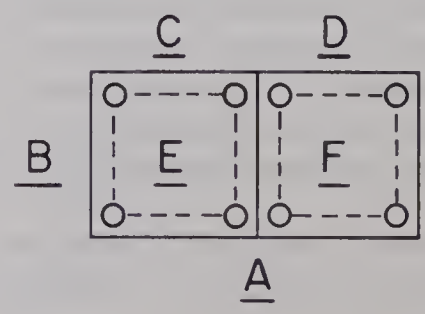
Based on the first example at the top of *Work Text* page 52, such discussions ought to preceed the rest of the page, taking care that everyone understands the sketch shorthand. If some one points out the inconsistency of the following:



as hardly logical shorthand for

We have to agree. But, we can persuade children to overlook this bit of stretching in the interests of convenience. As long as we admit such a special case as a bit on the odd side, it will no longer concern students.

In our shorthand sketch we have used numbers in 6 locations, and agreed on their meaning:



A number located at B indicates rows of objects in each of the two sections or arrays; C indicates columns in one section or number of objects in each row; D indicates the same for the other section; E indicates total number of objects in one section; and F the total number in the other section; A indicates total number of dots in both sections together.

Betty's games on *Work Text* page 52 place certain limitations on the number that can be used for some of these aspects of an arrangement. In example *b*, the rules of the game, in effect, require that there be a multiple of 7 dots in each row of the array on the left and less than 7 dots in each row of the array on the right. We are also told that there are 50 objects in all, arranged in 5 rows; so, there must be 10 to a row. Our choice for a number in each row of the left section is between 0×7 and 1×7, because $2 \times 7 = 14$ and 5×14 is more than 50. If we select 0×7, then there must be 10 in each row of the array in the section on the right. But the rules require that this number be less than 10. So, we are left with a choice of 1×7 or 7 in each row on the left—and this dictates 3 on the right because $10 - 7 = 3$ or $3 + 7 = 10$.

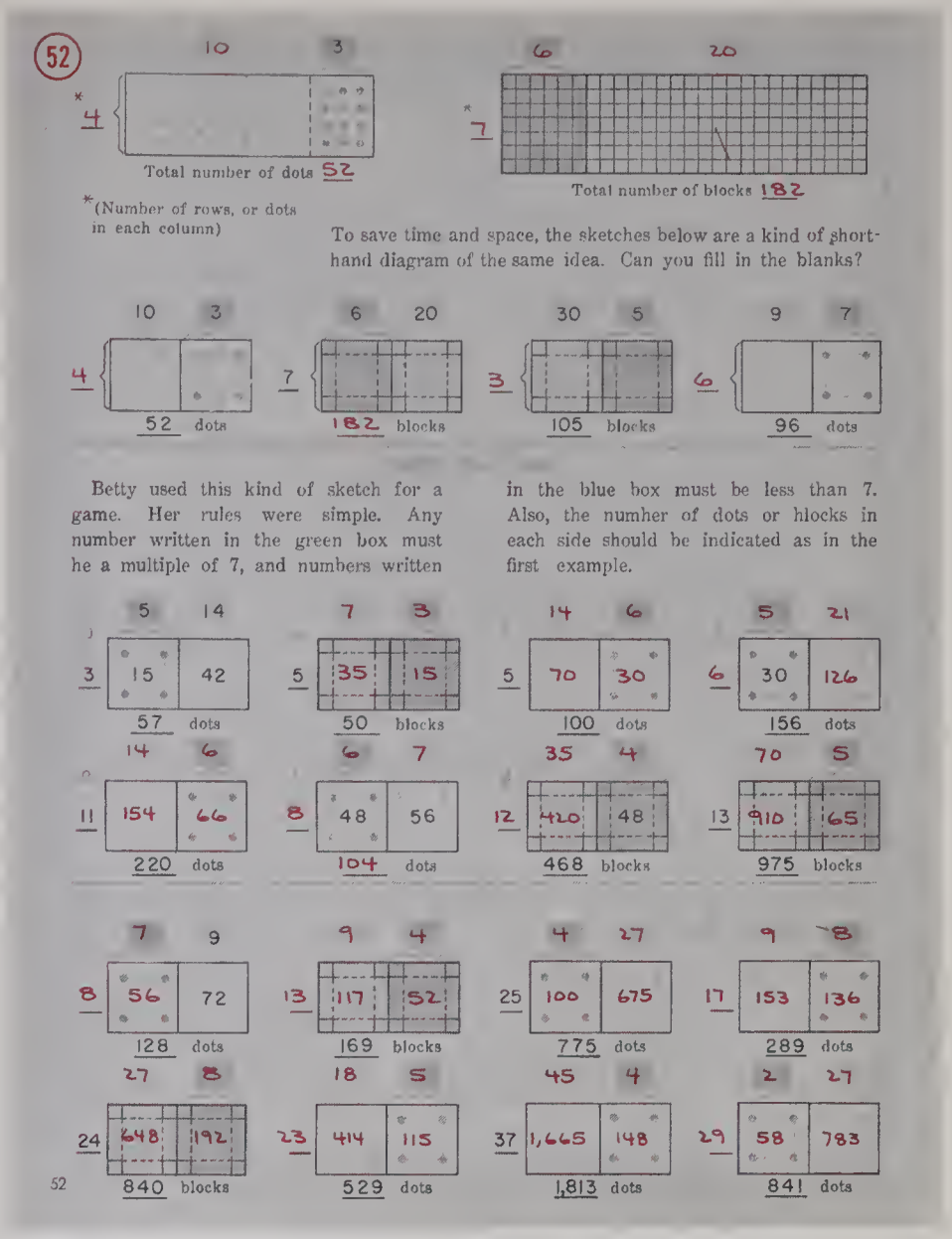
The rules change and the information given changes, but the general approach to solving the problems is the same. In each case, often for subtle reasons, there is but a single answer that meets the conditions given and prescribed by rule. (In the last example—*p*—it would be helpful if the solver first considered possible factors of 841: they are 29 and 29—and some others except 1, which is excluded generally, though not specifically here, unless 1 must be used as a factor.)

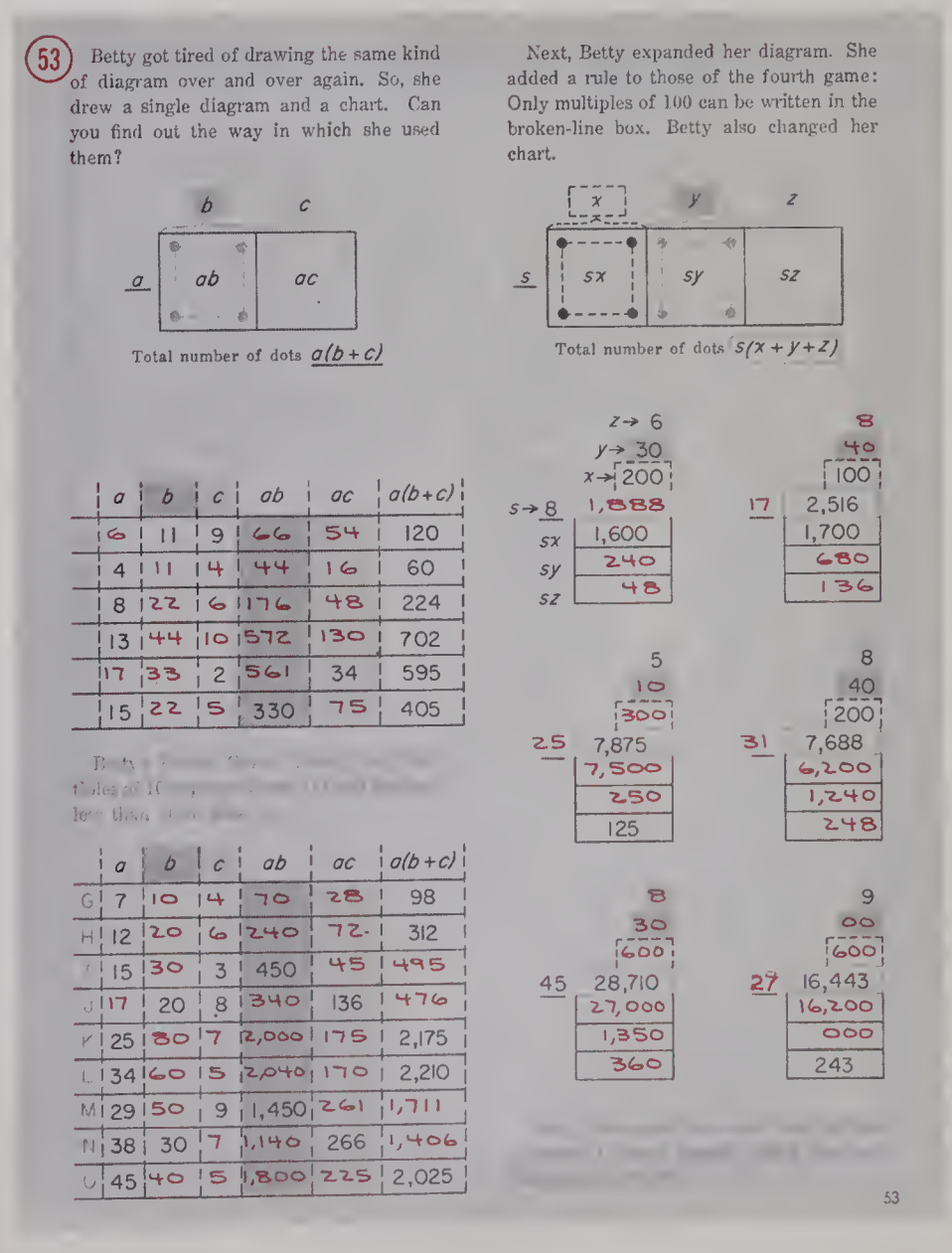
On *Work Text* page 53, the game employs more sketch shorthand—a general sketch, together with a chart for each rule based on the idea behind the sketch. Again, be sure that this shorthand is thoroughly understood. It will be used time and again to point to what's going on—and becomes one very powerful way to help children understand algebraic multiplication later on. If the shorthand used here and on the next page is understood, then, from *Work Text* page 54 it is clear that:

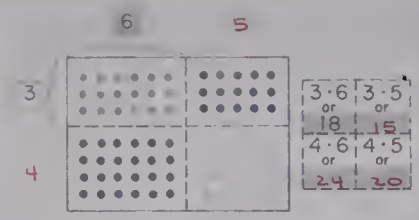
$$(a+b)\,(c+d) = ac + ad + bc + bd$$

and that is an idea from algebra that in the past has been a big hurdle to students. Because they were not familiar with this idea in arithmetic, it was strange when met in their algebra class.

Further, this sketch shorthand helps to explain what goes on in the multiplication algorism:

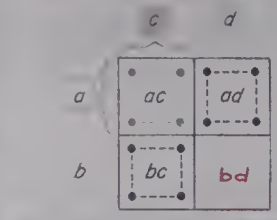

52
Total number of dots 52
(Number of rows, or dots in each column)
Total number of blocks 182
To save time and space, the sketches below are a kind of shorthand diagram of the same idea. Can you fill in the blanks?
52 dots
182 blocks
105 blocks
96 dots
Betty used this kind of sketch for a game. Her rules were simple. Any number written in the green box must he a multiple of 7, and numbers written in the blue box must be less than 7. Also, the numher of dots or hlocks in each side should be indicated as in the first example.
57 dots
50 blocks
100 dots
156 dots
220 dots
104 dots
468 blocks
975 blocks
128 dots
169 blocks
775 dots
289 dots
840 blocks
529 dots
1,813 dots
841 dots
52


53
Betty got tired of drawing the same kind of diagram over and over again. So, she drew a single diagram and a chart. Can you find out the way in which she used them?
Total number of dots a(b+c)
Next, Betty expanded her diagram. She added a rule to those of the fourth game: Only multiples of 100 can be written in the broken-line box. Betty also changed her chart.
Total number of dots s(x+y+z)
53



18 + 15 + 24 + 20 = 77

A diagram and record of one example.



$ac + ad + bc + bd = (a+b)(c+d)$

A general diagram to be used for all examples in the chart below.

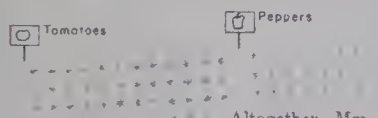
Study the example above of (3 + 4) (6 + 5) and its chart. A similar pattern can be seen in the general diagram at the right. Compare the letters with the chart below. The pattern in this diagram is used in all of the examples below. (No restrictions have been placed yet on the numbers that can be indicated in any of the boxes.)

(Draw diagrams of any examples that trouble you.)

	a	b	c	d	ac	ad	a(c+d)	bc	bd	b(c+d)	(a+b)(d+c)
I	3	4	6	5	18	15	33	24	20	44	77
II	6	4	8	2	48	12	60	32	8	40	100
III	5	8	10	5	50	25	75	80	40	120	195
IV	40	5	3	6	120	240	360	15	30	45	405
V	7	9	20	9	140	63	203	180	81	261	464
VI	20	4	30	7	600	140	740	120	28	148	888
VII	30	1	20	3	600	90	690	20	3	23	713

	a	b	c	d	ac	ad	a(c+d)	bc	bd	b(c+d)	(a+b)(c+d)
VIII	20	3	10	7	200	140	340	30	21	51	391
IX	30	4	50	6	1,500	180	1,680	200	24	224	1,904
X	20	7	20	5	400	100	500	140	35	175	675
XI	10	9	30	7	300	70	370	270	63	333	703
XII	40	3	40	3	1,600	120	1,720	120	9	129	1,849
XIII	110	4	10	8	1,100	880	1,980	40	32	72	2,052
XIV	50	5	30	7	1,500	350	1,850	150	35	185	2,035

In her garden, Mrs. Green set out 8 rows of plants. In each row, there were a dozen tomato plants and 7 pepper plants.

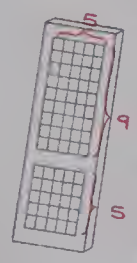


Altogether, Mrs. Green set out 152 plants. Of these, 96 were tomato plants and 56 were pepper plants. She set out 40 more tomato plants than pepper plants.

5·30 + 5·70 = 500

Mr. White wanted to put new panes of glass in a door. In the top section of the door there were 9 rows of 5 panes each. In the bottom section, 5 rows with 5 panes in each.

He needed a total of 70 panes; 45 for the top section and 25 for the bottom section.



25(28 + 42) = 1,750

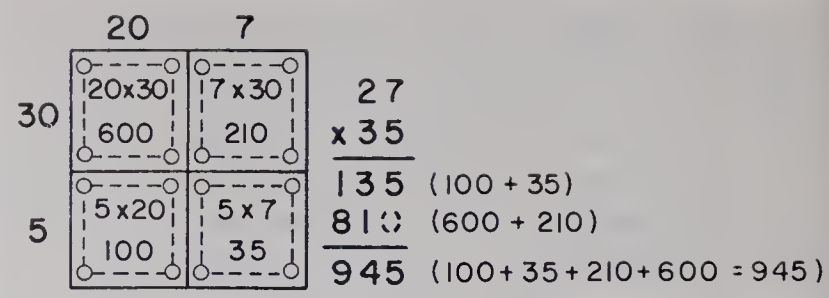
(15 + 17) 20 = 640

Mrs. Swanson bought 8 cans of peas at 23 cents each, and 8 cans of corn at 27 cents each. For the cans of peas and corn she paid a total of $ 4.00

15·25 + 18·25 = 825

				Total Amount of Purchases
8	23¢	8	27¢	$ 4.00

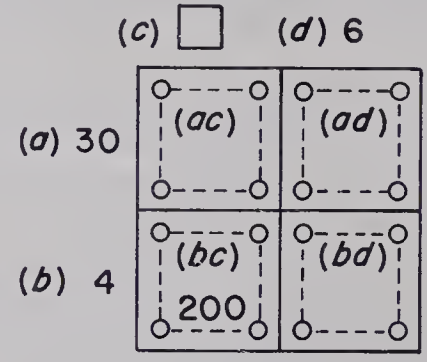
8(12 + 17 + 21) = 400



An expansion of the "two-array" sketch shorthand is explained in the second column on page 53: "a three-array" version.

The "2 by 2 version" of the sketch is introduced on *Work Text* page 54 by a particular example and then generalized so that all the rest of the examples on the page are based on it . . . certainly a convenient way to indicate 14 rather than with involved problems.

If any of the problems are confusing, draw a particular sketch for it. Consider example 9. Information given in the chart is shown, and the rule requires a multiple of 10 in the box.



If the bottom left array has a total of 200 objects arranged in 4 rows, there must be 50—which is a multiple of 10 and can be written in the box. Now the rest is straightforward computation.

Work Text page 55 should be given as much time as possible and enlarged upon in class activities. "Where do we find examples of the distributive principle pattern in the world of everyday experience?"

Three commonplace examples are given as thought starters.

"Might one use it as a shortcut in determining the number of seats in an auditorium?"

Consider a checkerboard. There are 8 rows, with 4 black spaces and 4 red in each row. The pattern could be expressed as:

$$8(4+4) = 32 + 32$$

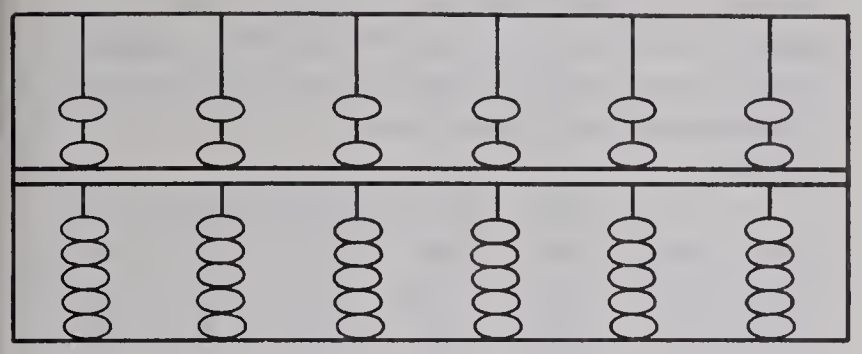
So, there are 32 red spaces and 32 black spaces, or

64 in all. Another way to express the pattern would be:

$$4 \cdot 8 + 4 \cdot 8 = 4(8+8) = 4 \cdot 16 = 64$$

There are 4 red in each of 8 rows and 4 blacks in each of 8 rows.

"How many beads on this abacus?"



$$6 \times 2 + 6 \times 5 = 6(2+5) = 6 \times 7 = 42$$

Notes to accompany *Work Text* pages 56-61

Many versions of the legend of the Grand Vizier have been passed down to us. The particulars given on page 56 vary from the classic in that the inventor of chess is usually credited with asking for only a single grain on the first square of the board. We have a more greedy Grand Vizier. The extra grain on the first square almost doubles his total reward, but since we find out the full measure can never be paid anyway, the greediness is of no consequence.

The reason for our slight twist to the classic story becomes evident as the activity on *Work Text* pages 56 and 57 develop. The first impression we hope the work makes is that some series of numbers start off with a seemingly modest rate of growth, and before they get very far the rate of growth has become amazingly rapid.

Consider the change from term to term in three series we have met:

A	0	3	6	9	12	15	18	21
change	3	3	3	3	3	3	3	
	0	0	0	0	0	0		

B	0	1	4	9	16	25	36	49
change	1	3	5	7	9	11	13	
	2	2	2	2	2	2		
	0	0	0	0	0			

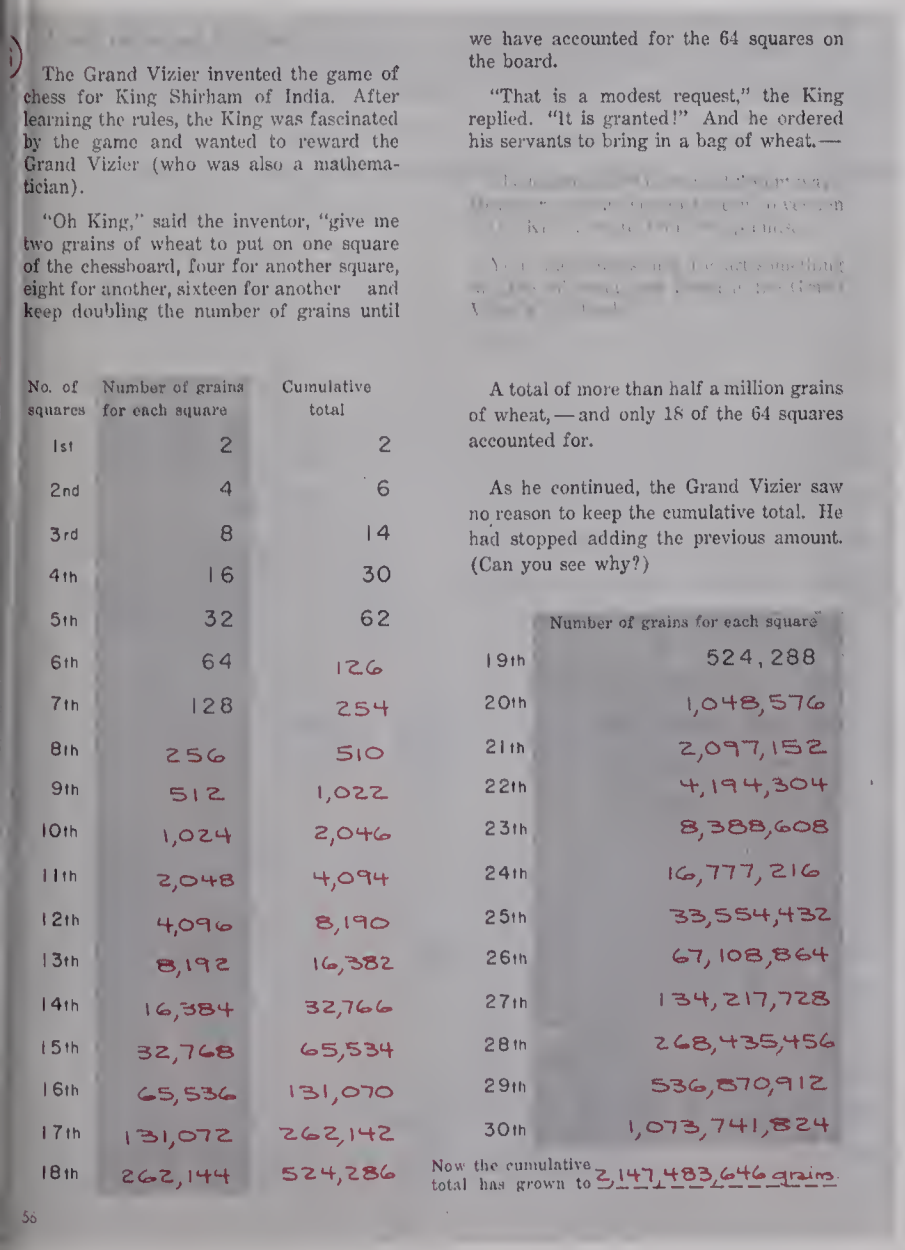
C	(1)	2	4	8	16	32	64	128
change	1	2	4	8	16	32	64	
	1	2	4	8	16	32		

A is the "multiplication table of 3" and has a constant change: each term is 3 more than its predecessor.

B is the table of squares: 0×0, 1×1, 2×2, 3×3, 4×4, etc. Each of its terms increases by the "next odd number."

C is a series in which each term is twice its predecessor—and the line of amount of difference simply repeat the line above it, the line of change.

When we consider the step-by-step increase in the amounts of increase—see second and third rows under each series—there is a definite difference of variety indicated:



The Grand Vizier invented the game of chess for King Shirham of India. After learning the rules, the King was fascinated by the game and wanted to reward the Grand Vizier (who was also a mathematician).

"Oh King," said the inventor, "give me two grains of wheat to put on one square of the chessboard, four for another square, eight for another, sixteen for another and keep doubling the number of grains until we have accounted for the 64 squares on the board.

"That is a modest request," the King replied. "It is granted!" And he ordered his servants to bring in a bag of wheat.—

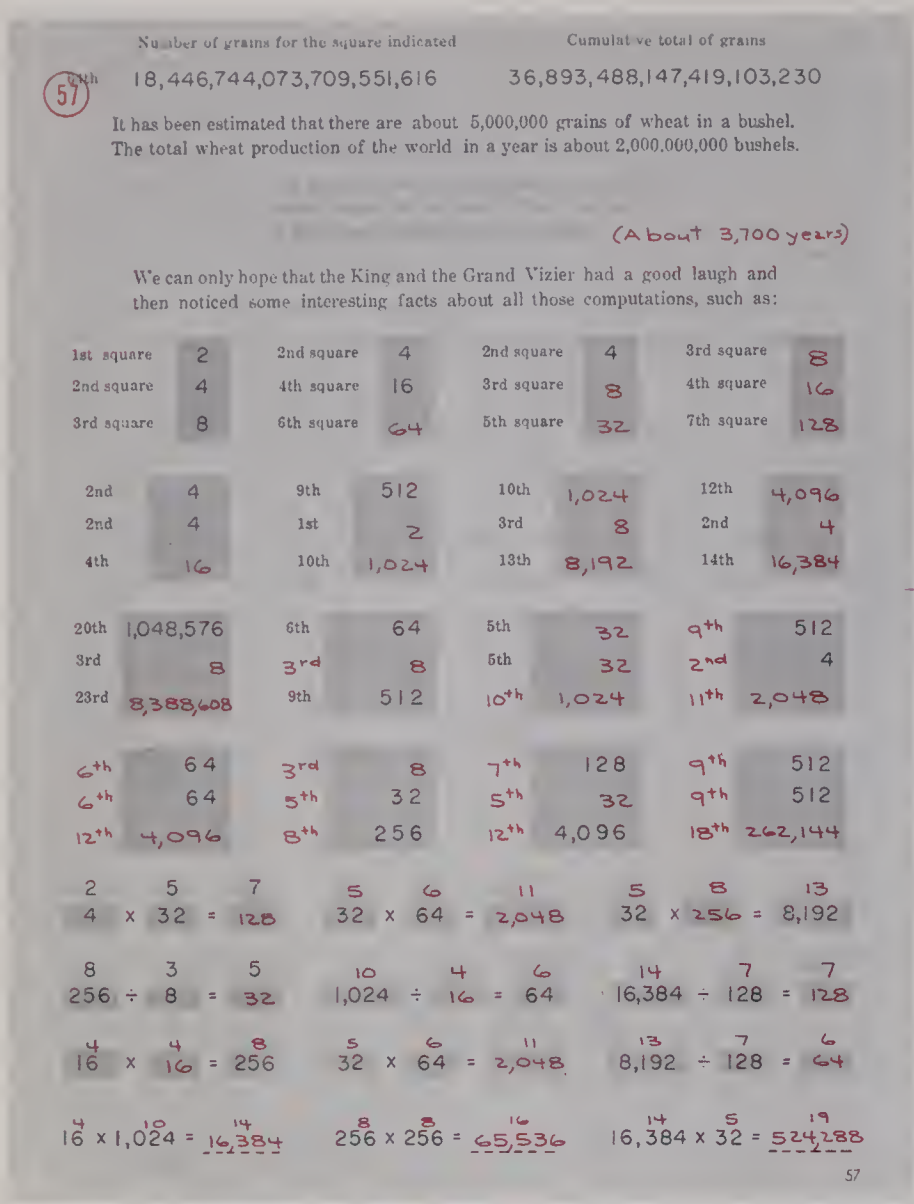
No. of squares	Number of grains for each square	Cumulative total
1st	2	2
2nd	4	6
3rd	8	14
4th	16	30
5th	32	62
6th	64	126
7th	128	254
8th	256	510
9th	512	1,022
10th	1,024	2,046
11th	2,048	4,094
12th	4,096	8,190
13th	8,192	16,382
14th	16,384	32,766
15th	32,768	65,534
16th	65,536	131,070
17th	131,072	262,142
18th	262,144	524,286

A total of more than half a million grains of wheat,—and only 18 of the 64 squares accounted for.

As he continued, the Grand Vizier saw no reason to keep the cumulative total. He had stopped adding the previous amount. (Can you see why?)

	Number of grains for each square
19th	524,288
20th	1,048,576
21th	2,097,152
22th	4,194,304
23th	8,388,608
24th	16,777,216
25th	33,554,432
26th	67,108,864
27th	134,217,728
28th	268,435,456
29th	536,870,912
30th	1,073,741,824

Now the cumulative total has grown to 2,147,483,646 grains.



A: By the second row, the amount of difference is 0.

B: By the third row, the amount of difference is 0.

C: The series keeps repeating itself and should never show a difference of 0.

A starts out fast, but is soon overtaken and passed by B. C starts out slowest of all, but soon outstrips the other two—and the Grand Vizier rapidly corners all the wheat in the world (or did he lose his head?).

Another aspect

A second aspect of this activity is a study of patterns in this rapidly growing series of numbers suggested by the Grand Vizier's reward formula. When he has reached the 18th square with its 262,144 the accumulated amount for those squares is 2 × 262,144 − 2; and he notices that this is a pattern that persisted from the lst square:

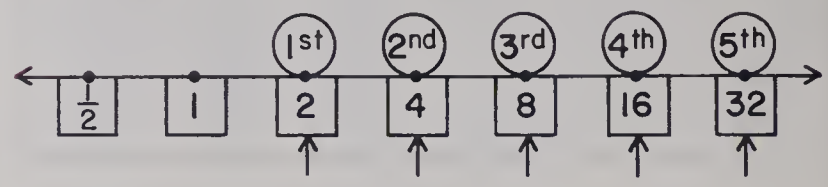
Squares	Grains	Accumulation
1st	2	2 or (2×2−2)
2nd	4	6 or (2×4−2)
3rd	8	14 or (2×8−2)
4th	16	30 or (2×16−2)

Once he recognized this pattern, it was a waste of time to compute it for all squares. When he reached the 30th square, his reward would be increased by 1,073,741,824 grains, so he found his cumulative total at that point as:

$$2 \times 1{,}073{,}741{,}824 - 2 = 2{,}147{,}483{,}646$$

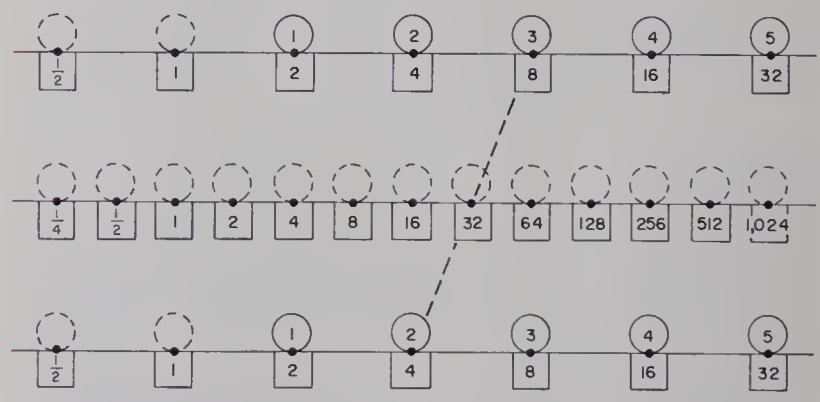
. . . already more than 2-billion grains of wheat.

The series generated by the Grand Vizier reward formula is a familiar one. Refer to the limited "multiplying machine" on *Work Text* page 34. Turned on its side (to conserve space here), one of its sides would look like this:

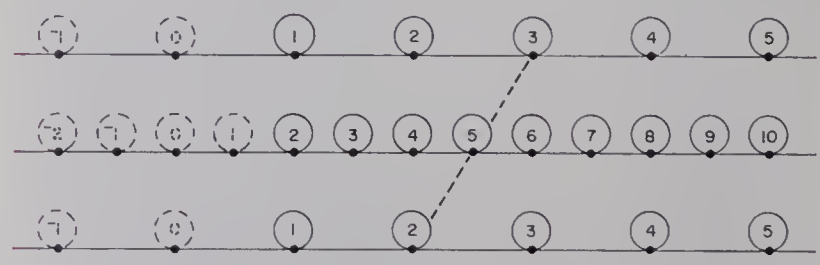


The arrows point to a series we find in the table on *Work Text* page 56—indicating the grains of wheat for each of the first five squares on the board. In the circles, the number of the square has been indicated.

Suppose we reconstruct the limited "multiplying machine" of *Work Text* page 34 (shown horizontally here) with both sides looking like the record above. We will leave the center or "product" line as it was. To refresh our memory of the machine's operation, we will find 4 on the bottom line and 8 on the top line. We note that the line passes through 32 on the center line. So, the machine tells us that 4 × 8 = 32, and we know the machine won't lead us astray.

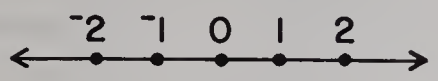


But we recognize a pattern in the number written in the circles—from an "adding machine" built on this same order:



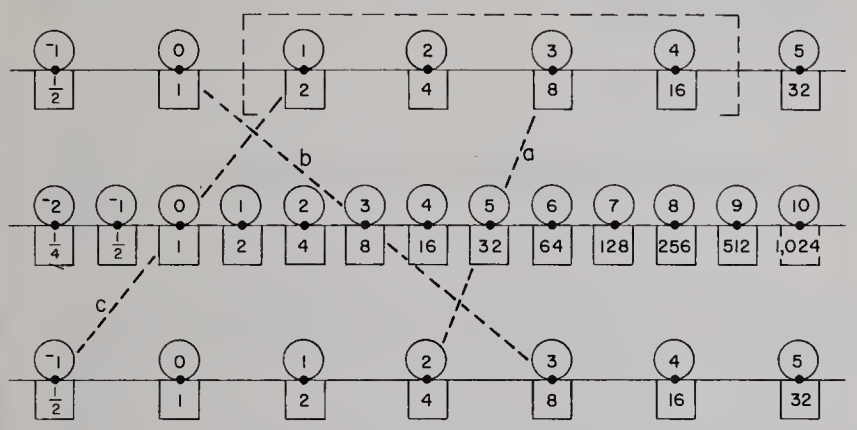
We locate 3 on the top line and 2 on the bottom line, and notice that it passes through 5 on the

middle line. So, the machine tells us that 3 + 2 = 5 . . . and we learn to rely on the machine to indicate sensible conclusions we can draw. Further, we can extend the pattern in the labels of each line—the pattern of labels on a number line like this:



Some children may be unfamiliar with −2 and −1, etc. Explain that this is very much like a scale on a thermometer: 2° below zero; 1° below zero; zero; 1° above zero; 2° above zero; etc. (If the class wants to discuss this further, look ahead to *Work Text* page 115 and amplify on the idea as fully as you want.)

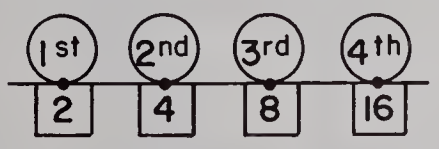
Now, it looks very much if we could combine our "multiplying machine" and "adding machine"—which can also be used as "dividing machines" and "subtracting machines"—into an all-purpose machine that looks like this:



Each of the lines drawn on the machine above tells us two things, and both are reasonable:

(a)	3 + 2 = 5	and	8 × 4 = 32
(b)	0 + 3 = 3	and	1 × 8 = 8
(c)	−1 + (+1) = 0	and	1/2 × 2 = 1

Recall the origin of our all-purpose machine. The relationship between the labels on the two machines was determined by the relationship between the number of the squares on the checkerboard and the number of grains of wheat associated with it:



Keep a large sketch of the all-purpose machine above for later reference. But, now, back to *Work Text* page 57 and a study the Grand Vizier might make of his chart showing the rewards he expected.

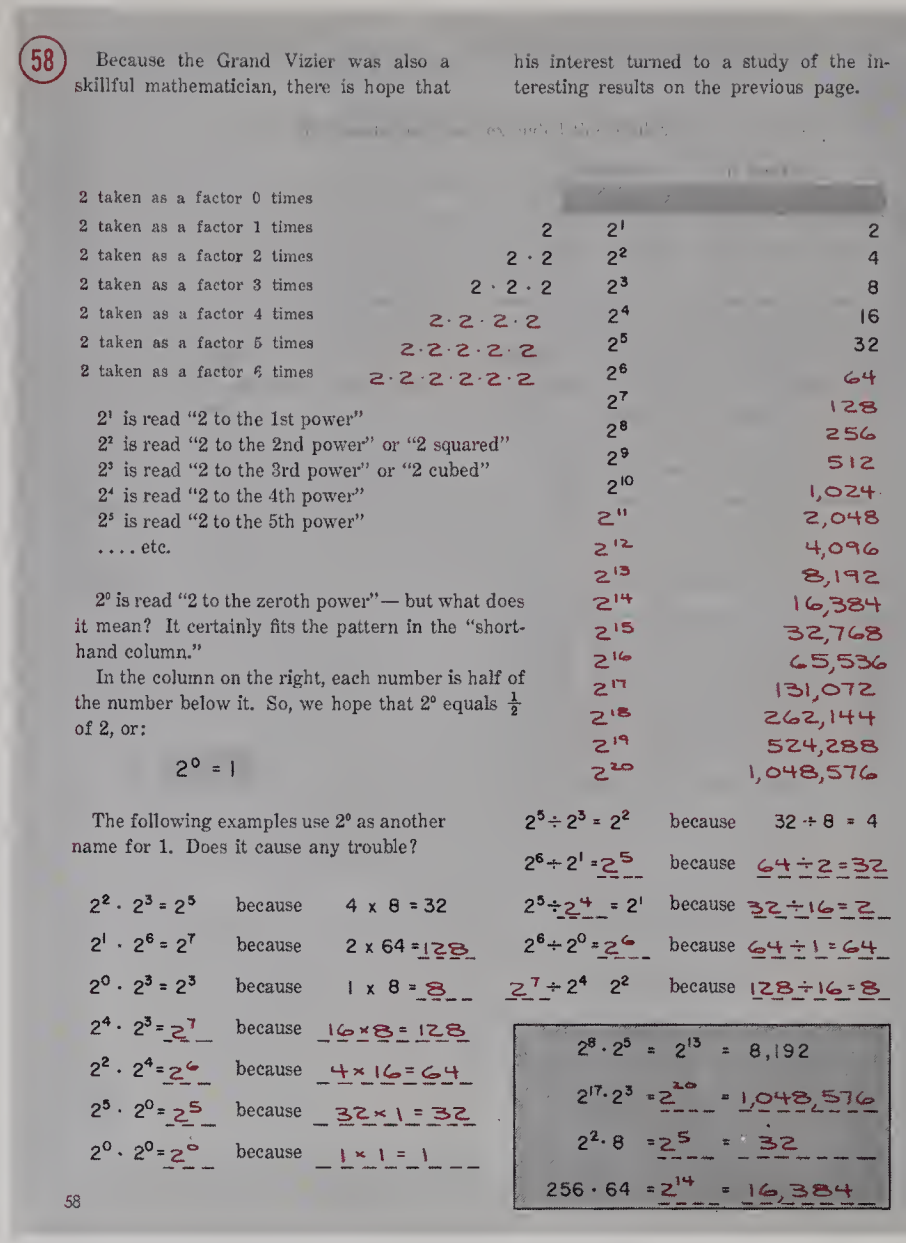
Try letting each child work his way through this page with a minimum of help. There is a powerful pattern revealed in these examples, one which is also an important pattern. These two arguments ought to convince anyone that this idea ought to be "discovered" by as many children as possible. If the pattern is not "discovered" as a student works through page 57, do not point it out to him—and do not despair, because he will confront it in a different context on *Work Text* page 58.

Work Text page 58, 59, 60 and the first column of page 61, are also designed for "discovery" through independent study. From time you time, you may find it profitable for two or three children to work together—but not for long. The danger is that some one will reveal the secret.

You may not have worked through pages 56-61 or considered carefully the pattern developed. You may be waiting for the secret to be revealed here. We are going to let you down. You, also, ought to be given the opportunity to discover the pattern.

For those who have become familiar with the relationship we have been exploring, you might like to discuss certain bypaths with all or some of the class.

For the "all-purpose" machine we notice some of the patterns in the following sequence:



58 Because the Grand Vizier was also a skillful mathematician, there is hope that his interest turned to a study of the interesting results on the previous page.

2 taken as a factor 0 times
2 taken as a factor 1 times 2
2 taken as a factor 2 times 2 · 2
2 taken as a factor 3 times 2 · 2 · 2
2 taken as a factor 4 times 2 · 2 · 2 · 2
2 taken as a factor 5 times 2 · 2 · 2 · 2 · 2
2 taken as a factor 6 times 2 · 2 · 2 · 2 · 2 · 2

2^1 is read "2 to the 1st power"
2^2 is read "2 to the 2nd power" or "2 squared"
2^3 is read "2 to the 3rd power" or "2 cubed"
2^4 is read "2 to the 4th power"
2^5 is read "2 to the 5th power"
.... etc.

2^0 is read "2 to the zeroth power"— but what does it mean? It certainly fits the pattern in the "shorthand column."

In the column on the right, each number is half of the number below it. So, we hope that 2^0 equals $\frac{1}{2}$ of 2, or:

$2^0 = 1$

2^1	2
2^2	4
2^3	8
2^4	16
2^5	32
2^6	64
2^7	128
2^8	256
2^9	512
2^{10}	1,024
2^{11}	2,048
2^{12}	4,096
2^{13}	8,192
2^{14}	16,384
2^{15}	32,768
2^{16}	65,536
2^{17}	131,072
2^{18}	262,144
2^{19}	524,288
2^{20}	1,048,576

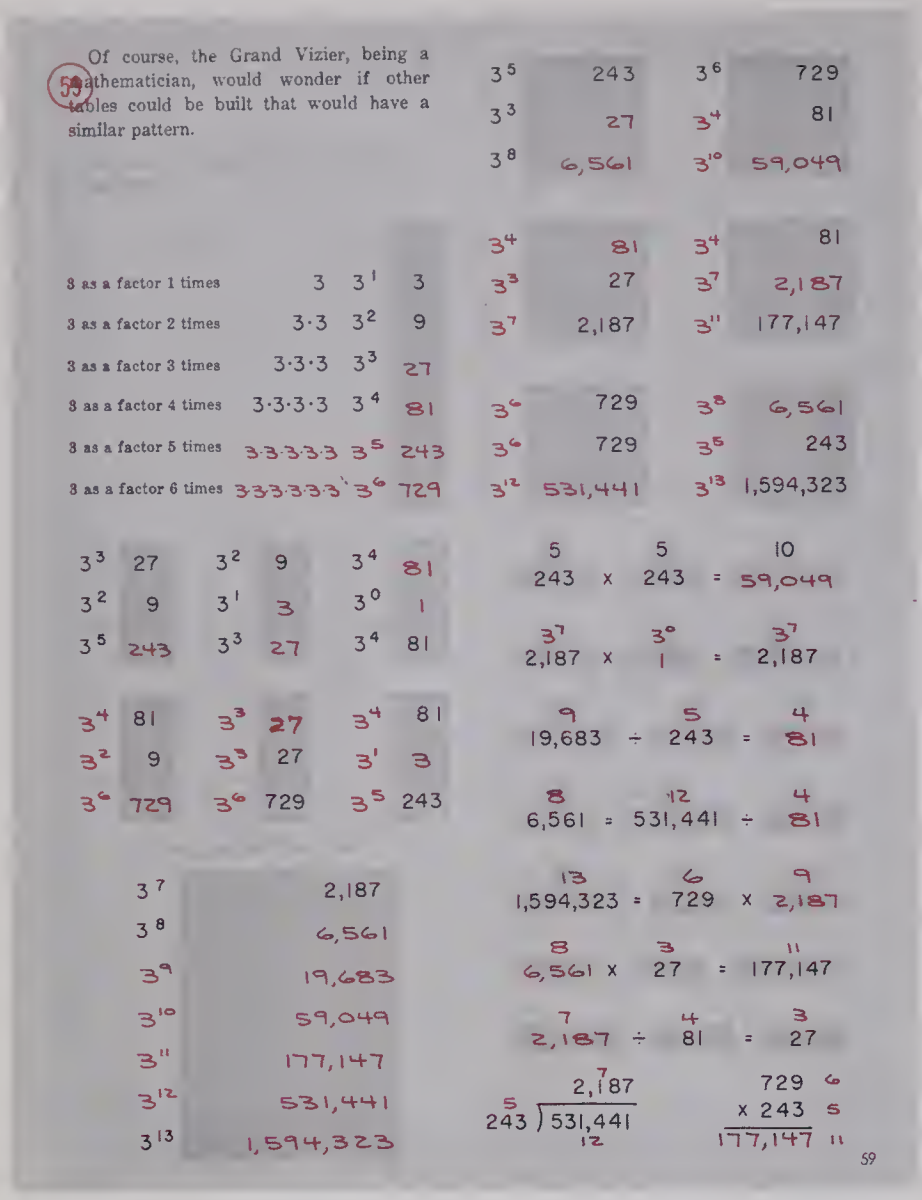
The following examples use 2^0 as another name for 1. Does it cause any trouble?

$2^2 \cdot 2^3 = 2^5$ because $4 \times 8 = 32$
$2^1 \cdot 2^6 = 2^7$ because $2 \times 64 = 128$
$2^0 \cdot 2^3 = 2^3$ because $1 \times 8 = 8$
$2^4 \cdot 2^3 = 2^7$ because $16 \times 8 = 128$
$2^2 \cdot 2^4 = 2^6$ because $4 \times 16 = 64$
$2^5 \cdot 2^0 = 2^5$ because $32 \times 1 = 32$
$2^0 \cdot 2^0 = 2^0$ because $1 \times 1 = 1$

$2^5 \div 2^3 = 2^2$ because $32 \div 8 = 4$
$2^6 \div 2^1 = 2^5$ because $64 \div 2 = 32$
$2^5 \div 2^4 = 2^1$ because $32 \div 16 = 2$
$2^6 \div 2^0 = 2^6$ because $64 \div 1 = 64$
$2^7 \div 2^4$ 2^2 because $128 \div 16 = 8$

$2^8 \cdot 2^5 = 2^{13} = 8{,}192$
$2^{17} \cdot 2^3 = 2^{20} = 1{,}048{,}576$
$2^2 \cdot 8 = 2^5 = 32$
$256 \cdot 64 = 2^{14} = 16{,}384$

58



Of course, the Grand Vizier, being a mathematician, would wonder if other tables could be built that would have a similar pattern.

3 as a factor 1 times	3	3^1	3
3 as a factor 2 times	3·3	3^2	9
3 as a factor 3 times	3·3·3	3^3	27
3 as a factor 4 times	3·3·3·3	3^4	81
3 as a factor 5 times	3·3·3·3·3	3^5	243
3 as a factor 6 times	3·3·3·3·3·3	3^6	729

3^3	27	3^2	9
3^2	9	3^1	3
3^5	243	3^3	27

3^4	81	3^3	27
3^2	9	3^3	27
3^6	729	3^6	729

3^4	81	3^4	81
3^2	9	3^1	3
3^5	243	3^5	243

3^7	2,187
3^8	6,561
3^9	19,683
3^{10}	59,049
3^{11}	177,147
3^{12}	531,441
3^{13}	1,594,323

3^5	243	3^6	729
3^3	27	3^4	81
3^8	6,561	3^{10}	59,049

3^4	81	3^4	81
3^3	27	3^7	2,187
3^7	2,187	3^{11}	177,147

3^6	729	3^8	6,561
3^6	729	3^5	243
3^{12}	531,441	3^{13}	1,594,323

$5 \quad 5 \quad 10$
$243 \times 243 = 59{,}049$

$3^7 \quad 3^0 \quad 3^7$
$2{,}187 \times 1 = 2{,}187$

$9 \quad 5 \quad 4$
$19{,}683 \div 243 = 81$

$8 \quad 12 \quad 4$
$6{,}561 = 531{,}441 \div 81$

$13 \quad 6 \quad 9$
$1{,}594{,}323 = 729 \times 2{,}187$

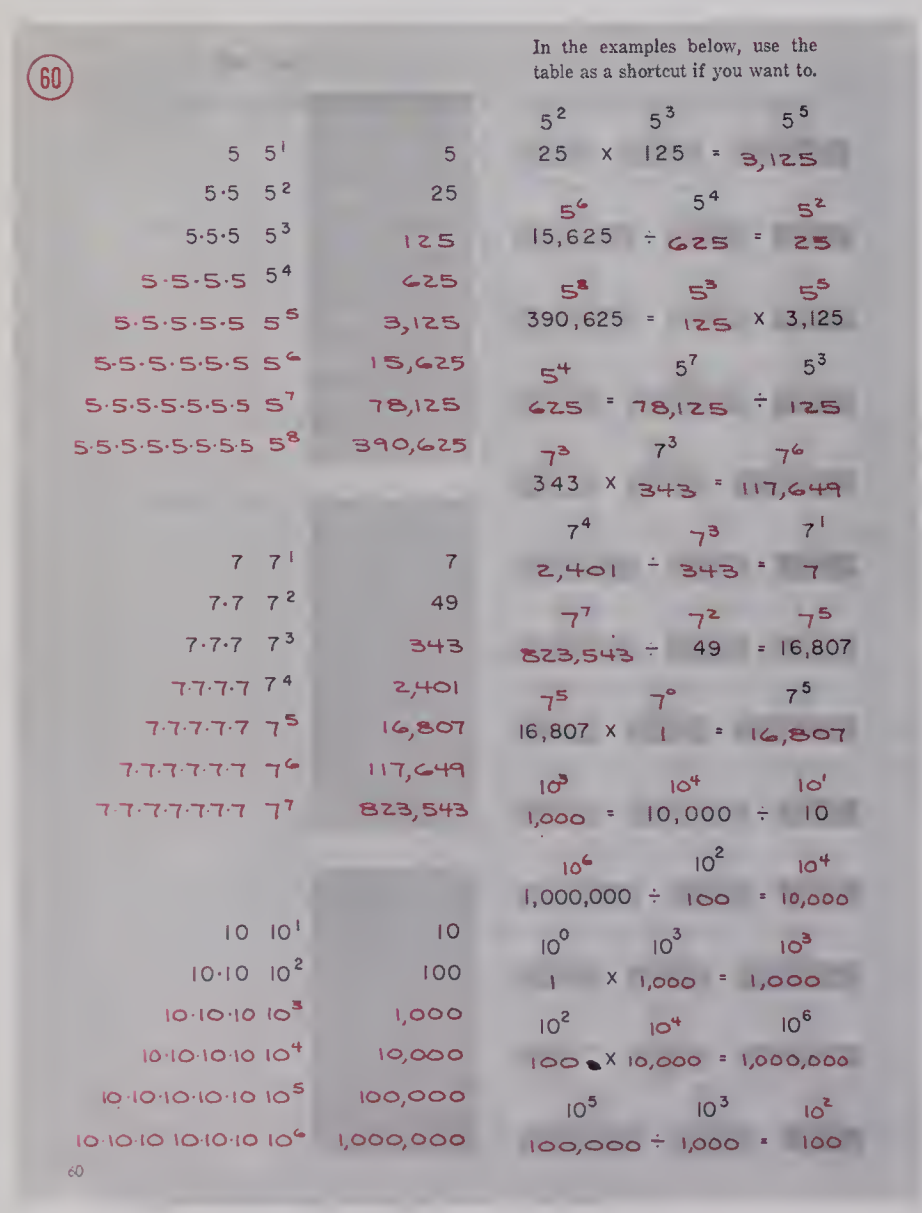
$8 \quad 3 \quad 11$
$6{,}561 \times 27 = 177{,}147$

$7 \quad 4 \quad 3$
$2{,}187 \div 81 = 27$

$$243 \overline{)531{,}441} \quad 2{,}187 \quad (5, 12, 7)$$

$$729 \times 243 = 177{,}147 \quad (6, 5, 11)$$

59



In the examples below, use the table as a shortcut if you want to.

5	5^1
5·5	5^2
5·5·5	5^3
5·5·5·5	5^4
5·5·5·5·5	5^5
5·5·5·5·5·5	5^6
5·5·5·5·5·5·5	5^7
5·5·5·5·5·5·5·5	5^8

5	5^1
25	5^2
125	5^3
625	5^4
3,125	5^5
15,625	5^6
78,125	5^7
390,625	5^8

7	7^1
7·7	7^2
7·7·7	7^3
7·7·7·7	7^4
7·7·7·7·7	7^5
7·7·7·7·7·7	7^6
7·7·7·7·7·7·7	7^7

7	7^1
49	7^2
343	7^3
2,401	7^4
16,807	7^5
117,649	7^6
823,543	7^7

10	10^1
10·10	10^2
10·10·10	10^3
10·10·10·10	10^4
10·10·10·10·10	10^5
10·10·10 10·10·10	10^6

5^2	25	5^3 × 125	5^5 = 3,125
5^6	15,625	5^4 ÷ 625	5^2 = 25
5^8	390,625	5^3 = 125	5^5 × 3,125
5^4	625	5^7 = 78,125	5^3 ÷ 125
7^3	343	7^3 × 343	7^6 = 117,649
7^4	2,401	7^3 ÷ 343	7^1 = 7
7^7	823,543	7^2 ÷ 49	7^5 = 16,807
7^5	16,807	7^0 × 1	7^5 = 16,807
10^3	1,000	10^4 = 10,000	10^1 ÷ 10
10^6	1,000,000	10^2 ÷ 100	10^4 = 10,000
10^0	1	10^3 × 1,000	10^3 = 1,000
10^2	100	10^4 × 10,000	10^6 = 1,000,000
10^5	100,000	10^3 ÷ 1,000	10^2 = 100

60

16	8	4	2	1	1/2	1/4	1/8	1/16
↓	↓	↓	↓	↓	↓	↓	↓	↓
2^4	2^3	2^2	2^1	2^0	___	___	___	___

Already, in the development on *Work Text* pages 58 and 59, we considered a meaning for $2^0 - 2^0 = 1$. That does not lead to any unfortunate conclusions, and it extends the pattern in the exponents. By the same kind of experimental approach, we might extend the pattern further so that:

2	1	1/2	1/4	1/8	1/16
↓	↓	↓	↓	↓	↓
2^1	2^0	2^{-1}	2^{-2}	2^{-3}	2^{-4}

What strange results follow!

If $2^1 = 2$ and $2^{-1} = 1/2$, we hope that $2^{-1} \times 2^1 = 2^{-1+1}$ or 2^0

Translated to using expressions free of exponents, we are saying:

$$2 \times 1/2 = 1,$$

which makes real sense. Let's try another.

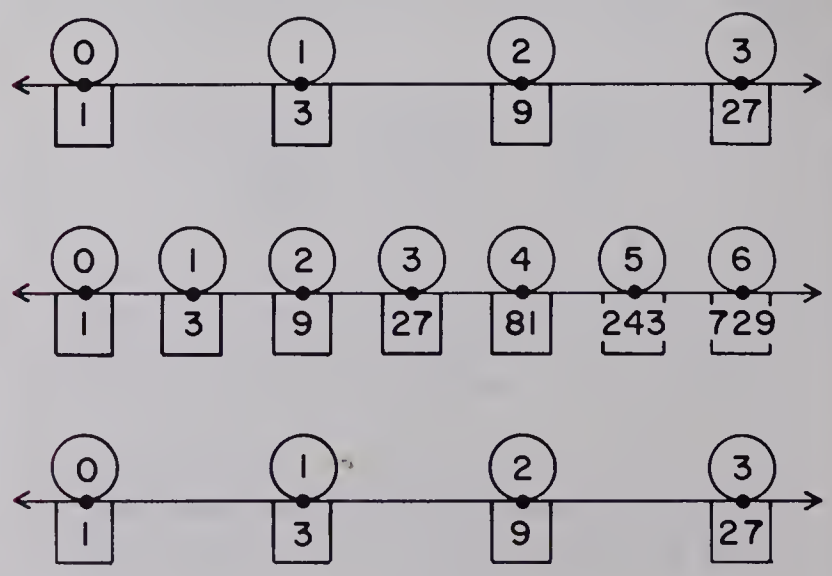
$$2^3 \times 2^{-2} = 2^{3+-2} = 2^1 = 2, \text{ or } 8 \times 1/4 = 2$$

Sensible again. Another way to look at this relationship:

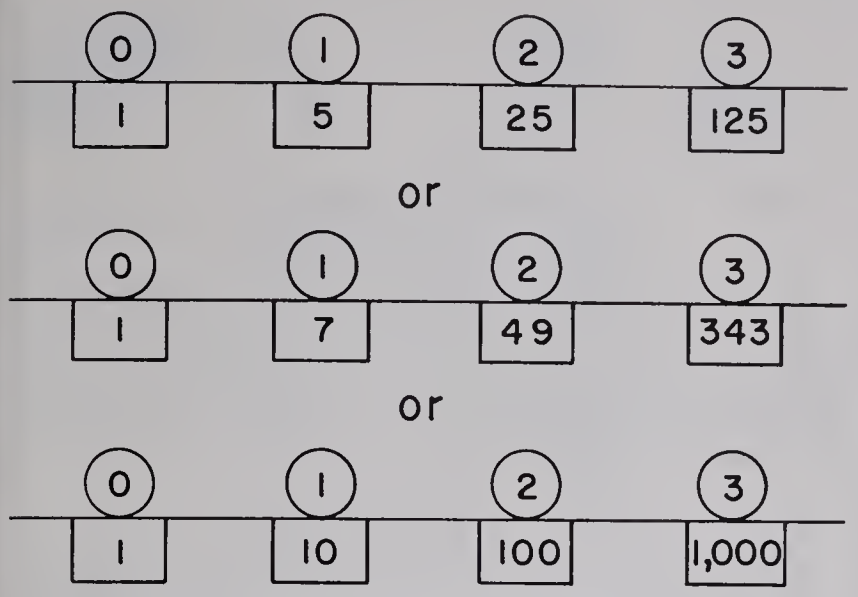
2^2	2^1	2^0	2^{-1}	2^{-2}	2^{-3}	2^{-4}
↓	↓	↓	↓	↓	↓	↓
4	2	1	$1/2^1$	$1/2^2$	$1/2^3$	$1/2^4$

But this extension is probably for a small group of any class at this time.

Another enrichment activity would be to build an all-purpose machine based on the development on *Work Text* page 59.



Others might build all-purpose machines based on the studies on *Work Text* page 60 in which the relationships are indicated by these initiating groups of labels:



Notes to accompany *Work Text* pages 62-69

The next eight pages obviously are concerned with important ideas of measurement, perimeters and areas, and scale drawing. But an equally important, and perhaps more basic, mathematical idea is developed at the same time. We will, at first, refer to it as "and-or."

The words "and" and "or" need to be carefully and exactly defined as they are used throughout mathematics. Their specific meaning there is a little different, certainly more precise, than their use in everyday discourse.

Let us define these terms with a series of examples: How many of the following words contain the letter A? How many contain the letter T? How many contain the letter A *and* the letter T? How many contain the letter A *or* the letter T?

MAN	ATE
HIM	THE
CUT	ASK
BAT	BET

We will list the groups we are interested in:

A: MAN; *BAT; ATE;* ASK (4)
T: CUT; *BAT; ATE;* THE; BET (5)
A *and* T: BAT; ATE (2)
A *or* T: MAN; CVT; BAT; ATE; THE; ASK; BET

61

Astronomers have watched stars that are

miles distant from earth. But writing all those zeros is a waste of time. For 2,500,000,000,000,000, scientists write

This is a great time saver. Work through the next group of examples. You can see the pattern of their system . It depends on an extension and further use of the table at the bottom of the opposite page.

$1{,}000 = 10 \cdot 10 \cdot 10 = 10^3$

$7{,}000 = 7 \cdot 10 \cdot 10 \cdot 10 = 7 \cdot 10^3$

$21{,}000 = 21 \cdot 10 \cdot 10 \cdot 10 = 21 \cdot 10^3$

$10{,}000 = 10 \cdot 10 \cdot 10 \cdot 10 = 10^4$

$80{,}000 = 8 \cdot 10\,10 \cdot 10 \cdot 10 = 8 \cdot 10^4$

$620{,}000 = 62\,10 \cdot 10 \cdot 10 \cdot 10 = 62 \cdot 10^4$

$100{,}000 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^5$

$50 \cdot 1{,}000 = 5 \cdot 10^1 \cdot 10^3 = 5 \cdot 10^4$

$100 \cdot 1{,}000 = 10^2 \cdot 10^3 = 10^5$

$400 \cdot 1{,}000 = 4 \cdot 10^2 \cdot 10^3 = 4 \cdot 10^5$

$1{,}200 \cdot 1{,}000 = 12 \cdot 10^2 \cdot 10^3 = 12 \cdot 10^5$

$3 \cdot 10^2 \cdot 5 \cdot 10^4 = 3 \cdot 5 \cdot 10^2 \cdot 10^4 = 15 \cdot 10^6$

$7 \cdot 10^4 \cdot 8 \cdot 10^7 = 7 \cdot 8 \cdot 10^4 \cdot 10^7 = 56 \cdot 10^{11}$

$3 \cdot 10^4 \cdot 7{,}000 = 3 \cdot 7 \cdot 10^4 \cdot 10^3 = 21 \cdot 10^7$

$85{,}000 \cdot 10{,}000 = 85 \cdot 10^3 \cdot 10^4 = 85 \cdot 10^7$

$3{,}000 \cdot 800 = 3 \cdot 8 \cdot 10^3 \cdot 10^2 = 24 \cdot 10^5$

$900 \cdot 110{,}000 = 9 \cdot 11 \cdot 10^2 \cdot 10^4 = 99 \cdot 10^6$

$10^2 \cdot 10^5 \cdot 10^4 = 10^{11}$

Our earth is about 93,000,000 miles from the sun, or

The furthest planet from the sun, Pluto, is about 40 times farther away.

Saturn is only about 10 times as far from the sun as our earth is — or $10 \times 93{,}000{,}000$ miles.

$93 \cdot 10^7$

Before they used radio telescopes, astronomers could see only about 30,000,000,000 stars, or

10^{10}

One of the largest they could see has a diameter of 2,400,000,000 miles, or

10^8

Our moon varies its distance from the earth from between 221,000 to 253,000 miles. When it is 250,000 miles away, we write this as:

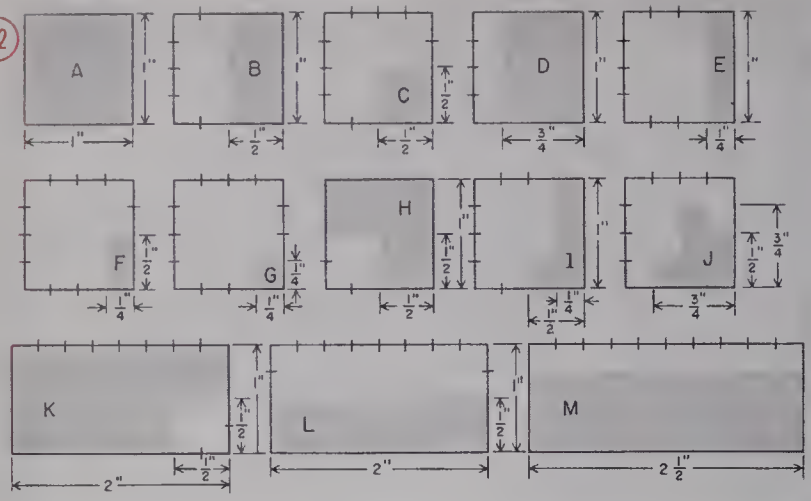
10^4

Uranus, another planet, has four moons. Uranus is 1,800,000,000 miles from our sun, or

$18 \cdot 10^8$

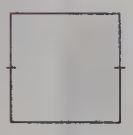
How many atoms are there in the Universe ? — Approximately
300,000,000,000,000

62



	A	B	C	D	E	F	G	H	I	J	K	L	M
S →	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{3}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$1\frac{3}{4}$	1	$1\frac{7}{8}$
U →	0	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$	$\frac{1}{4}$	$\frac{11}{16}$	$\frac{5}{8}$	$\frac{1}{4}$	1	$\frac{5}{8}$

S → number of square inches in shaded area; U → number of square inches in unshaded area.



What is the size, in square inches, of the area? . . .

. . . inside the black line? 1
. . . inside the green line? 2
. . . inside the black, but not inside the green lines? $\frac{1}{2}$
. . . inside the green, but not inside the black lines? $1\frac{1}{2}$
. . . inside both black and green lines? $\frac{1}{2}$ (blue tint)
. . . inside either black or green lines? $2\frac{1}{2}$ (entire sketch)
. . . inside either black or green lines, but not in black and green? 2 (all except blue)

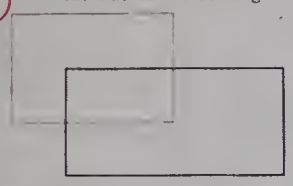
On tbe next page, you will need to remember this particular way we are using the words "and" and "or."

We will use "and" to express tbe idea of "in both" or in "all three" at the same time.

We will use "or" to express the idea of "in either" of two or "in any" of all three or more. Keep your eye on those words.

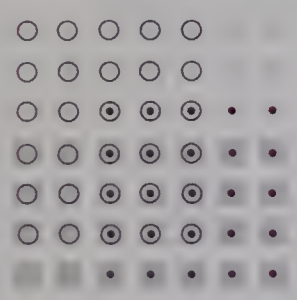
63

Use a ruler to determine the size of the rectangles.



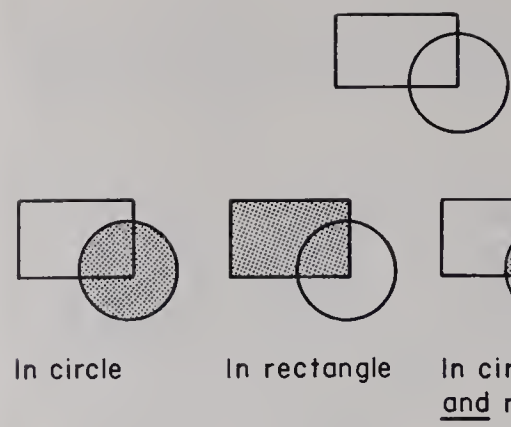
Hints

(a)	inside the green line?	$1\frac{1}{2}$	(d)	in either green or black?	3
(b)	inside the black line?	2	(e)	in either black or blue?	$2\frac{3}{4}$
(c)	inside the blue line?	1	(f)	in both green and black?	$\frac{1}{2}$
			(g)	in both black and blue?	$\frac{1}{4}$
			(h)	in green but not in black?	1
			(i)	in blue but not in black?	$\frac{3}{4}$
			(j)	in black but not in green or blue?	$1\frac{1}{4}$



How many . . .	No.
blocks? (green or blue)	49
green blocks?	28
blue blocks?	21
circles? (with or without dots)	30
dots? (with or without circles)	25
blocks without dots or circles?	6

	No.
blue blocks with dots?	5
blue blocks with circles?	15
blue blocks with dots or circles?	17
blue blocks with dots and circles?	3
green blocks with dots?	20
green blocks with circles?	15
green blocks with dots or circles?	26
green blocks with dots and circles?	9
blocks (green or blue) with dots or circles?	37
blocks with dots and circles?	12
blue blocks without dots?	16
green blocks without circles?	13
blue blocks without circles?	6
green blocks without dots and circles?	19



In circle | In rectangle | In circle and rectangle | In circle or rectangle

Another example: List all whole number multiples of 3 (including 0) that are less than 25. We will call that List D. List all whole number multiples of 2 (including 0) that are less than 25. We will call that List F.

List D: 0, 3, 6, 9, 12, 15, 18, 21, 24
List F: 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24
List all the numbers that are in List D *and* F: 0, 6, 12, 18
List all the numbers that are in List D *or* F: 0, 2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24.

Another example: (1) Think of all of the odd numbers less than 100. (2) Think of all the whole number multiples of 7 (including 0) that are less than 100.

How many odd numbers less than 100? 50

How many whole number multiples of 7 (including 0) less than 100? 15

How many belong to the first list *and* the second list? 7 (7, 21, 35, 49, 63, 77, 91).

How many belong to the first list *or* the second list? 58 (All 50 odd numbers plus 0, 14, 28, 42, 56, 70, 84, 98).

Many examples similar to those above can be used to increase facility in handling the basic facts of arithmetic, such as:

List A: All numbers less than 50 that leave a remainder of 2 when divided by 6.

List B: All numbers less than 100 that leave a remainder of 4 when divided by 7.

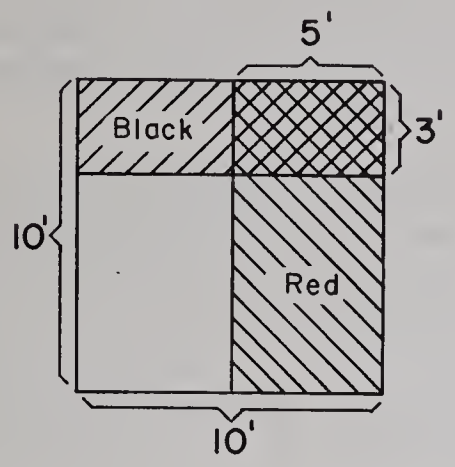
How many in List A? (List them: .....)

How many in List B? (List them: .....)

How many in A *and* B? (List them:.....)

How many in A *or* B? (List them:.....)

Here is another example to help define "and" and "or" as these two words are used in mathematics. Below is a sketch of a room that is 10′ by 10′. A strip, 3′ wide, was painted black along one side. Then a strip, 5′ wide, was painted red along an adjacent side.



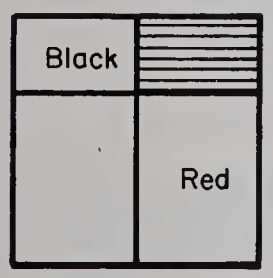
What is the area of the room? 100 square feet (10′ × 10′)

What is the area that was painted black? 30 sq. ft. (3′ × 10′)

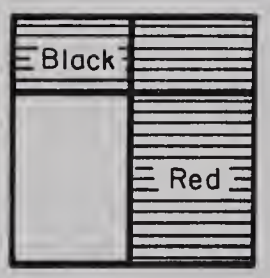
What is the area that was painted red? 50 sq. ft. (5′ × 10′)

What is the area that was painted black *and* red? 15 sq. ft. (3′ × 5′)

What is the area that was painted black *or* red? 45 sq. ft.



black AND red

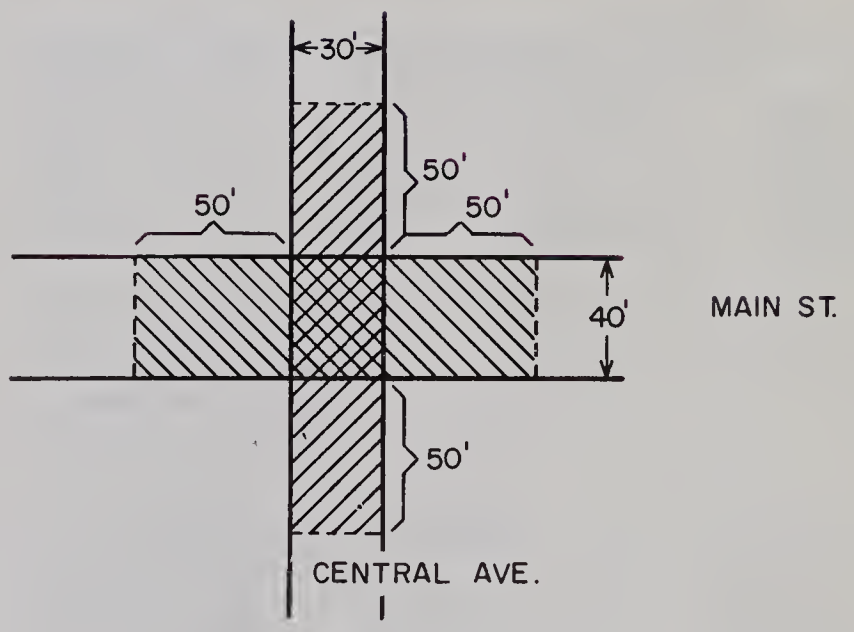


black OR red

"And" suggests that ideas as "overlap," "both at once"—the "intersection" of areas or groups or lists.

"Or" suggest such ideas as "altogether," either one or both"—the "union" of areas or groups or lists.

This language suggests another example: Ace Contractors were awarded a contract to replace the paving where Main Street crosses Central Avenue—to a distance of 50 feet from the corner in all 4 directions. Main Street was 40 feet wide and Central Avenue was 30 feet wide. Here is a sketch:

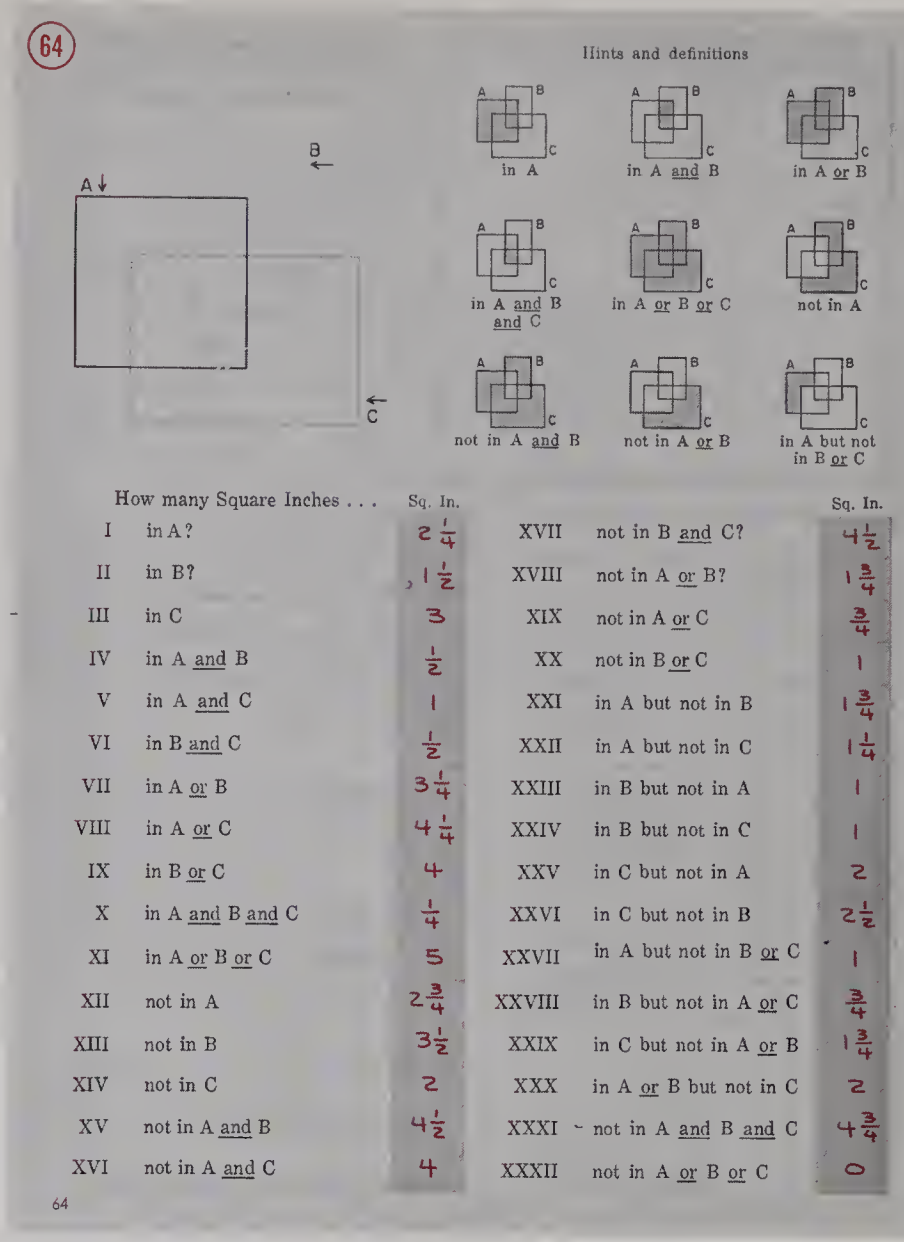


How much paving was laid on Main Street? 5,200 sq. ft. (40′ × 130′)

How much was repaved on Central Avenue? (5,200 sq. ft. (30′ × 140′)

How much paving was replaced in the "intersection?" 1,200 sq. ft. (30′ × 40′)

How much paving was replaced altogether? How much in the "union" of all the area covered by the contract? 9,200 square feet.


64

Hints and definitions

	How many Square Inches . . .	Sq. In.			Sq. In.
I	in A?	$2\frac{1}{4}$	XVII	not in B and C?	$4\frac{1}{2}$
II	in B?	$1\frac{1}{2}$	XVIII	not in A or B?	$1\frac{3}{4}$
III	in C	3	XIX	not in A or C	$\frac{3}{4}$
IV	in A and B	$\frac{1}{2}$	XX	not in B or C	1
V	in A and C	1	XXI	in A but not in B	$1\frac{3}{4}$
VI	in B and C	$\frac{1}{2}$	XXII	in A but not in C	$1\frac{1}{4}$
VII	in A or B	$3\frac{1}{4}$	XXIII	in B but not in A	1
VIII	in A or C	$4\frac{1}{4}$	XXIV	in B but not in C	1
IX	in B or C	4	XXV	in C but not in A	2
X	in A and B and C	$\frac{1}{4}$	XXVI	in C but not in B	$2\frac{1}{2}$
XI	in A or B or C	5	XXVII	in A but not in B or C	1
XII	not in A	$2\frac{3}{4}$	XXVIII	in B but not in A or C	$\frac{3}{4}$
XIII	not in B	$3\frac{1}{2}$	XXIX	in C but not in A or B	$1\frac{3}{4}$
XIV	not in C	2	XXX	in A or B but not in C	2
XV	not in A and B	$4\frac{1}{2}$	XXXI	not in A and B and C	$4\frac{3}{4}$
XVI	not in A and C	4	XXXII	not in A or B or C	0

64

Why is the "union" or the total area not 5200 sq. feet + 5200 sq. ft.? Because they paved the intersection only once.

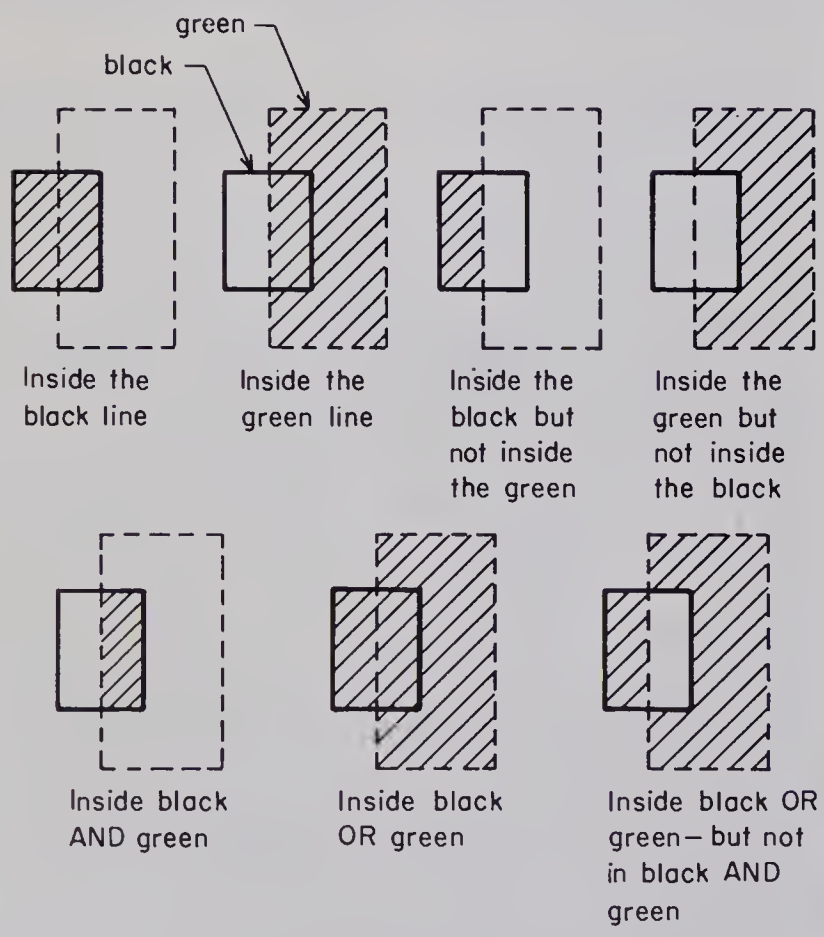
All earlier examples were developed to build an understanding of the specific meaning of "and" and "or." Only after such extensive exploration of the idea will students begin to see a common thread – the idea of "and" and the idea or "or" – the ideas that mathematicians refer to as "intersection" and "union."

It is best to reserve any formal discussion of the words "and" and "or" until the concept is well established. Then these words can be considered as they have been in the last example:

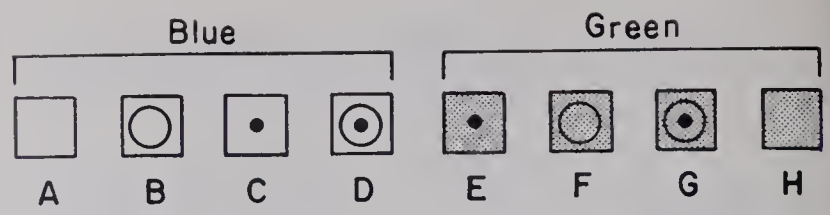
"AND":"both at once," "overlap" – "Intersection"
"OR":"either one or both," "altogether" – "Union"

With some such introductory work, the development on *Work Text* pages 62-69 will move ahead with little confusion.

As each question at the bottom of page 69 is considered, draw a sketch to be sure everyone understands the problem:


green
black
Inside the black line
Inside the green line
Inside the black but not inside the green
Inside the green but not inside the black
Inside black AND green
Inside black OR green
Inside black OR green – but not in black AND green

This same kind of study is extended at the top of *Work Text* page 63, but here a little more complex. The exercise in the lower portion of page 63 might well be introduced by another example to help define the ideas of "and" and "or."



How many blue blocks with dots? Obviously, the design labeled C is one that fits the conditions of the question. How about D? We are going to accept it also because we mean:

"How many blue blocks can answer 'yes' when asked, 'do you contain a dot?'" The design would answer "Yes" to that question in spite of the fact that it also contains a circle.

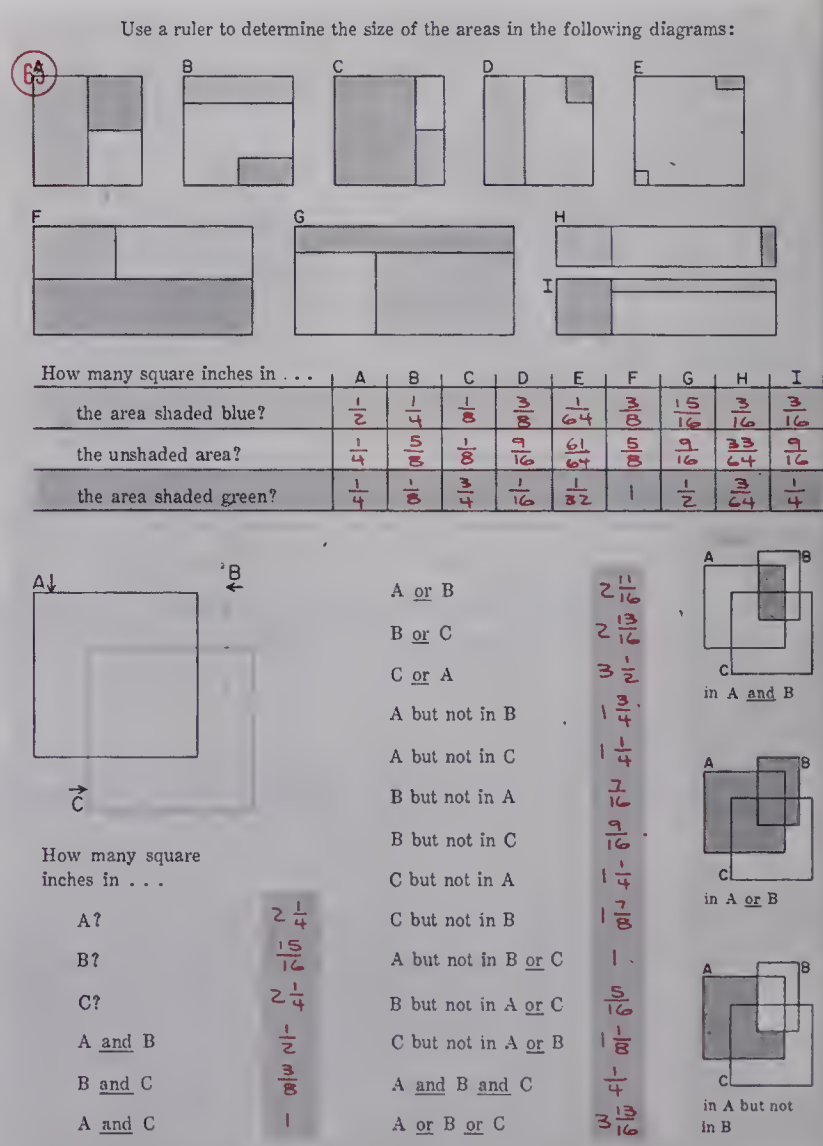
A further question: How many blocks are without dots *or* without circles?

Suppose each design is asked a pair of questions:

(1) Are you without a dot?
(2) Are you without a circle?

In answer to the first question, how many could answer "yes?" (A, B, F, H)

In answer to the second question, how many could answer "yes?" (A, C, E, H)


Use a ruler to determine the size of the areas in the following diagrams:

63

How many square inches in . . .	A	B	C	D	E	F	G	H	I
the area shaded blue?	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{64}$	$\frac{3}{8}$	$\frac{15}{16}$	$\frac{3}{16}$	$\frac{3}{16}$
the unshaded area?	$\frac{1}{4}$	$\frac{5}{8}$	$\frac{1}{8}$	$\frac{9}{16}$	$\frac{61}{64}$	$\frac{5}{8}$	$\frac{9}{16}$	$\frac{33}{64}$	$\frac{9}{16}$
the area shaded green?	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{4}$	$\frac{1}{16}$	$\frac{1}{32}$	1	$\frac{1}{2}$	$\frac{3}{64}$	$\frac{1}{4}$

How many square inches in . . .

A?	$2\frac{1}{4}$
B?	$\frac{15}{16}$
C?	$2\frac{1}{4}$
A and B	$\frac{1}{2}$
B and C	$\frac{3}{8}$
A and C	1

A or B	$2\frac{11}{16}$
B or C	$2\frac{13}{16}$
C or A	$3\frac{1}{2}$
A but not in B	$1\frac{3}{4}$
A but not in C	$1\frac{1}{4}$
B but not in A	$\frac{7}{16}$
B but not in C	$\frac{9}{16}$
C but not in A	$1\frac{1}{4}$
C but not in B	$1\frac{7}{8}$
A but not in B or C	1
B but not in A or C	$\frac{5}{16}$
C but not in A or B	$1\frac{1}{8}$
A and B and C	$\frac{1}{4}$
A or B or C	$3\frac{13}{16}$

65

How many answer "yes" to *either* one *or* the other or both? (A, B, C, E, F, H, – 6) So, 6 designs are without dots *or* without circles.

Had the original question been, "How many are without dots *and* without circles?", then we would find out how many would answer yet to both the first *and* the second questions: A, H (2). Only two designs could answer yes to both questions.

This discussion is necessary as explanation for such work as the last item in the left-hand column on *Work Text* page 63. The full question would be: "How many blocks are without a dot or without a circle? On page 63, the answer would be 37, because 37 of the designs could answer "Yes" to at least one of those questions. We can also say that only 12 of the 49 would be unable to answer yes to one question or the other.

Work Text pages 64 and 65 extend the idea of "and" and "or" to an arrangement in which three areas overlap in a certain way. The "limits" on each page help develop the definition. If, in any example, a student is confused, have him draw a sketch, shade in the area under consideration and note the needed dimensions.

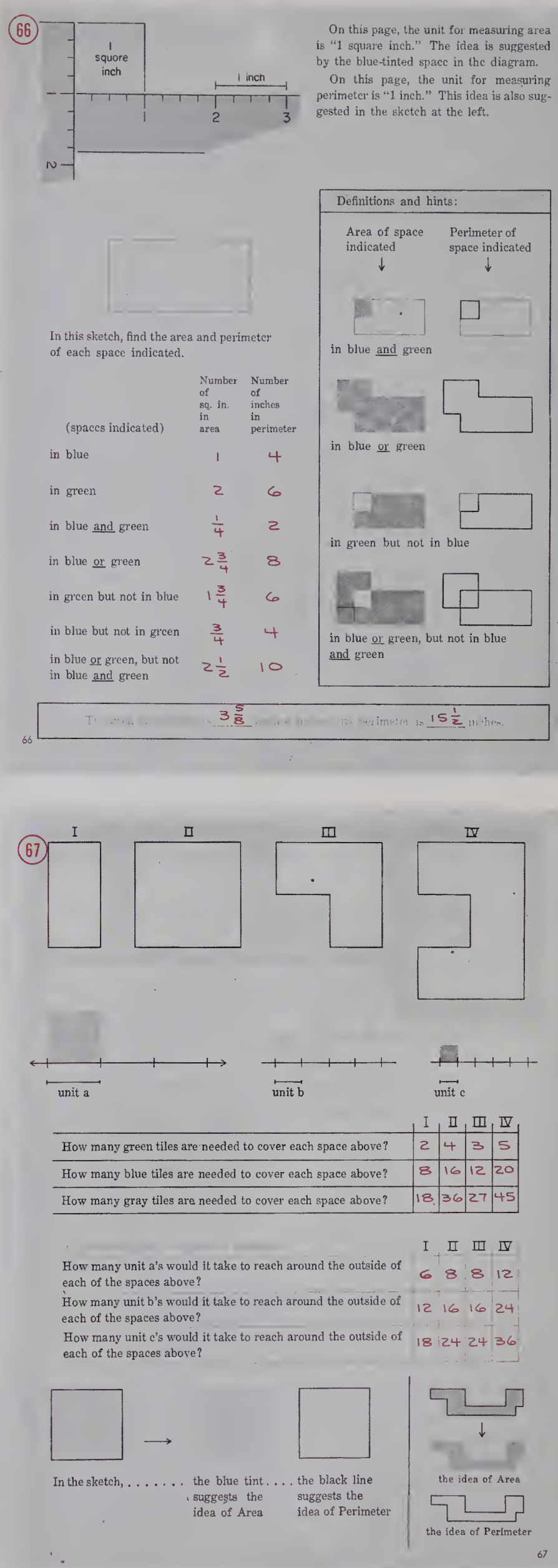
Pages 66 and 67 introduce the ideas of area and perimeter. The difficulty children and adults have in distinguishing these two measures of an area has long been with us. Galileo complained he met many people who did not have a good grasp on the difference between area and perimeter and continually confused the measure of the size of an area and the measure of the distance around its periphery.

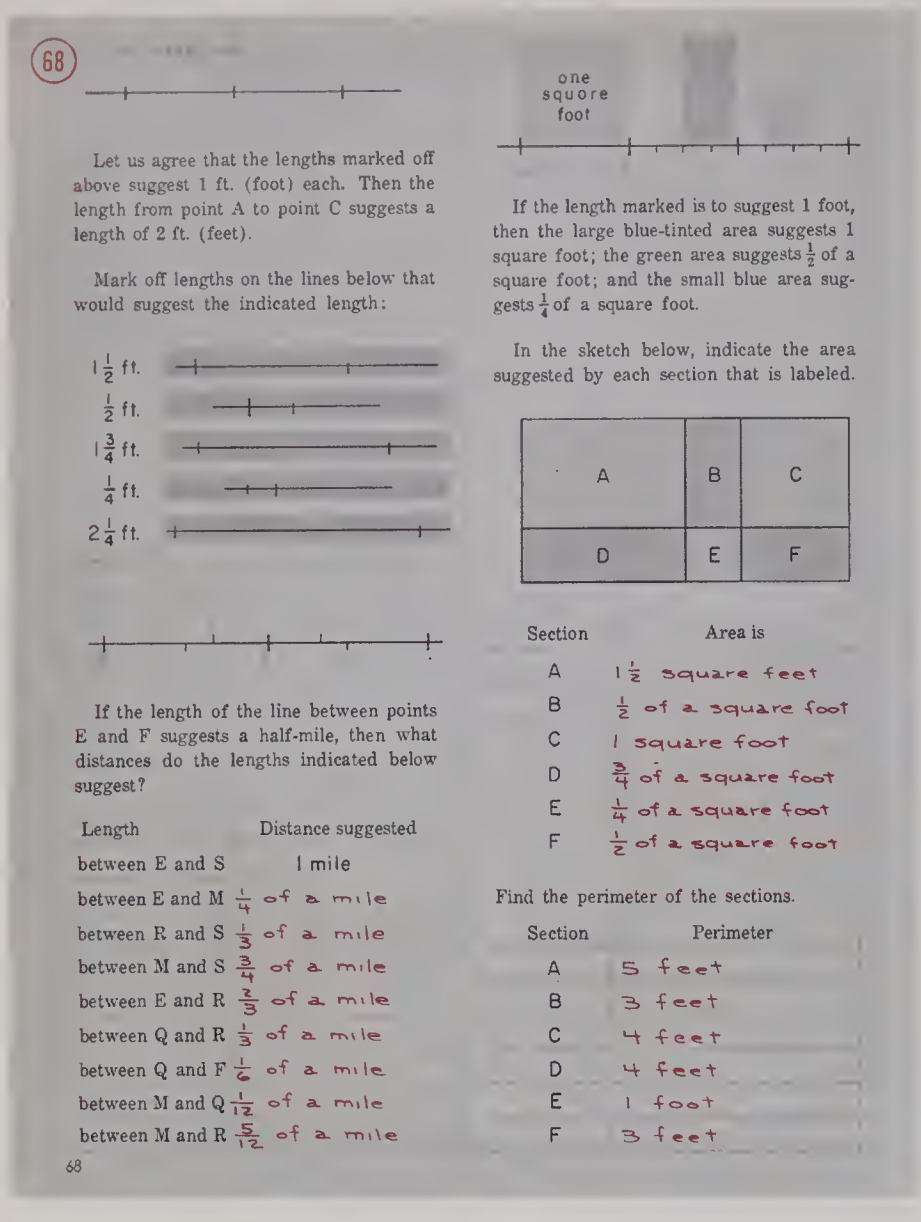
Measurement

Page 67 presents four sketches, two of them somewhat irregular. Three different systems of measurement are indicated by selection of 3 different units called *a*, *b*, and *c*. Each is a unit of linear measurement. A "tile" in each system is defined as a square that is one unit of measure along each side. Area, then, is expressed as the number of "tiles" required to cover each sketch completely and without overlapping.

The essential idea is that, once a unit of length is defined and a unit of area is defined (as the "tile" is defined), we can find the perimeter and the area in terms of that unit. This concept is basic to understanding "scale drawings" on pages 68 and 69, to the consideration to follow part of the metric system, to reading maps, etc.

In the first chart on *Work Text* page 69, certain


66
1 squore inch
1 inch
1
2
3
2
On this page, the unit for measuring area is "1 square inch." The idea is suggested by the blue-tinted space in the diagram.
On this page, the unit for measuring perimeter is "1 inch." This idea is also suggested in the sketch at the left.
In this sketch, find the area and perimeter of each space indicated.
(spaces indicated)
Number of sq. in. in area
Number of inches in perimeter
in blue 1 4
in green 2 6
in blue and green 1/4 2
in blue or green 2 3/4 8
in green but not in blue 1 3/4 6
in blue but not in green 3/4 4
in blue or green, but not in blue and green 2 1/2 10
Definitions and hints:
Area of space indicated
Perimeter of space indicated
in blue and green
in blue or green
in green but not in blue
in blue or green, but not in blue and green
3 5/8
15 1/2 inches
66
67
I
II
III
IV
unit a
unit b
unit c
How many green tiles are needed to cover each space above? 2 4 3 5
How many blue tiles are needed to cover each space above? 8 16 12 20
How many gray tiles are needed to cover each space above? 18 36 27 45
How many unit a's would it take to reach around the outside of each of the spaces above? 6 8 8 12
How many unit b's would it take to reach around the outside of each of the spaces above? 12 16 16 24
How many unit c's would it take to reach around the outside of each of the spaces above? 18 24 24 36
In the sketch, ... the blue tint . . . . suggests the idea of Area
the black line suggests the idea of Perimeter
the idea of Area
the idea of Perimeter
67



68

Let us agree that the lengths marked off above suggest 1 ft. (foot) each. Then the length from point A to point C suggests a length of 2 ft. (feet).

Mark off lengths on the lines below that would suggest the indicated length:

$1\frac{1}{2}$ ft.

$\frac{1}{2}$ ft.

$1\frac{3}{4}$ ft.

$\frac{1}{4}$ ft.

$2\frac{1}{4}$ ft.

If the length of the line between points E and F suggests a half-mile, then what distances do the lengths indicated below suggest?

Length	Distance suggested
between E and S	1 mile
between E and M	$\frac{1}{4}$ of a mile
between R and S	$\frac{1}{3}$ of a mile
between M and S	$\frac{3}{4}$ of a mile
between E and R	$\frac{2}{3}$ of a mile
between Q and R	$\frac{1}{3}$ of a mile
between Q and F	$\frac{1}{6}$ of a mile
between M and Q	$\frac{1}{12}$ of a mile
between M and R	$\frac{5}{12}$ of a mile

one square foot

If the length marked is to suggest 1 foot, then the large blue-tinted area suggests 1 square foot; the green area suggests $\frac{1}{2}$ of a square foot; and the small blue area suggests $\frac{1}{4}$ of a square foot.

In the sketch below, indicate the area suggested by each section that is labeled.

A	B	C
D	E	F

Section	Area is
A	$1\frac{1}{2}$ square feet
B	$\frac{1}{2}$ of a square foot
C	1 square foot
D	$\frac{3}{4}$ of a square foot
E	$\frac{1}{4}$ of a square foot
F	$\frac{1}{2}$ of a square foot

Find the perimeter of the sections.

Section	Perimeter
A	5 feet
B	3 feet
C	4 feet
D	4 feet
E	1 foot
F	3 feet

68

relationships should be noted and discussed. Without reference to the generating activities, put the following on the board:

	I	II	III	IV
a.	2	4	3	5
b.	8	16	12	20
c.	18	36	27	45
d.	10			
e.			9	
f.		32		
g.				
h.				

If the relationships are noted in the first three lines, then that pattern could be extended to complete the next three lines:

d.	10	20	15	25
e.	6	12	9	15
f.	16	32	24	40

And, in lines g and H, sequences of 4 numbers each can be suggested that fit the same pattern. How about:

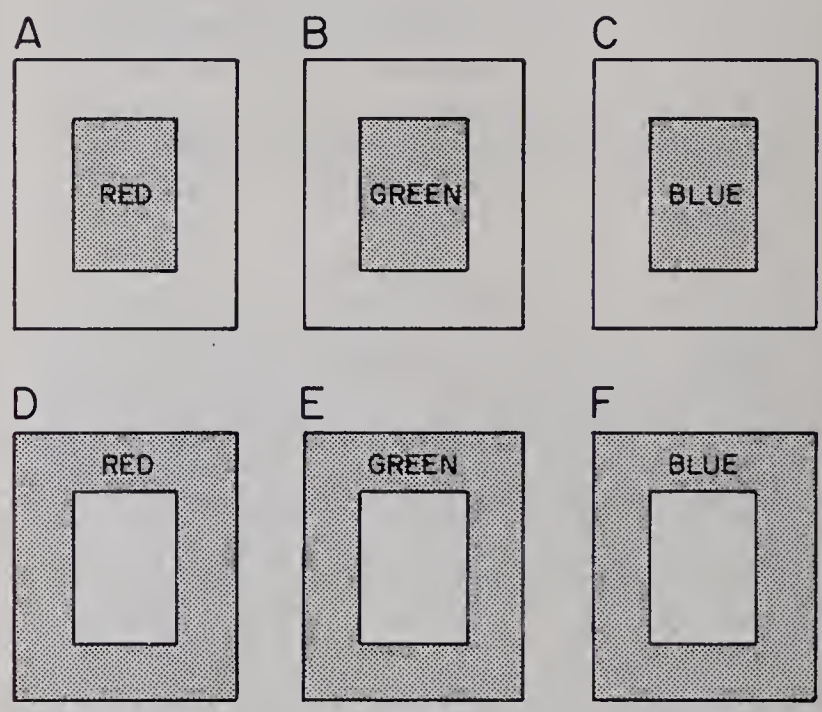
1, 2, 1 1/2, 2 1/2, or 2/5, 4/5, 3/5, 1, or 1/2, 1, 3/4, 1 1/4 ?

Then go back to the generating activity. "Why did this relationship develop?" It expresses the relative areas.

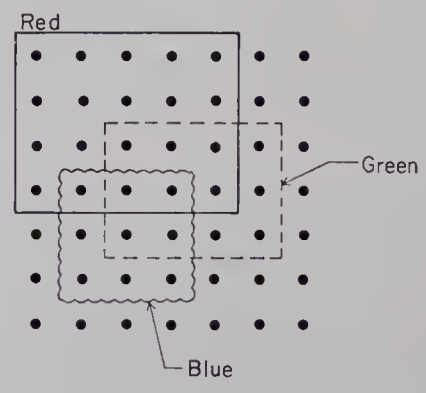
"What might lines *d* through *h* on the extended chart mean?" They might reflect results of choosing other basic units.

Page 68 considers the linear scales: "one inch represents one foot," and "three inches represents one mile." The activity is then extended to a combination of measurements based on the former scale for area as well as perimeter.

Page 69 combines the idea of drawings made to scale; the ideas of area and perimeter; and the ideas that have been defined for "and" and "or." Some children can learn to play a game that might be called, "AND-OR." Prepare a set of cards that look like this:



These cards get their meaning from a playing board that can be one of a great variety of designs, such as:



Card A: ... the number of dots *in* red ($4 \times 5 = 20$)

Card B: ... the number of dots *in* green ($3 \times 4 = 12$)

Card C: ... the number of dots *in* blue ($3 \times 3 = 9$)

Card D: ... the number of dots *not in* red ($49 - 20 = 29$)

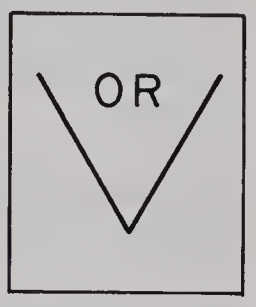
Card E: ... the number of dots *not in* green ($49 - 12 = 37$)

Card F: ... the number of dots *not in blue* ($49 - 9 = 40$)

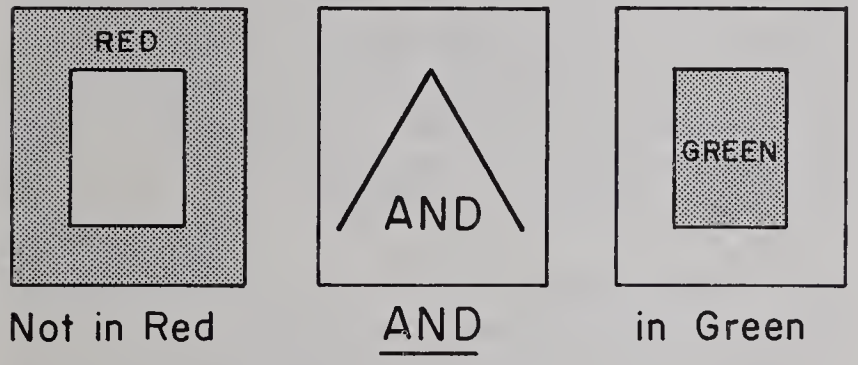
Two other cards (preferably several of each) are required:

G

H



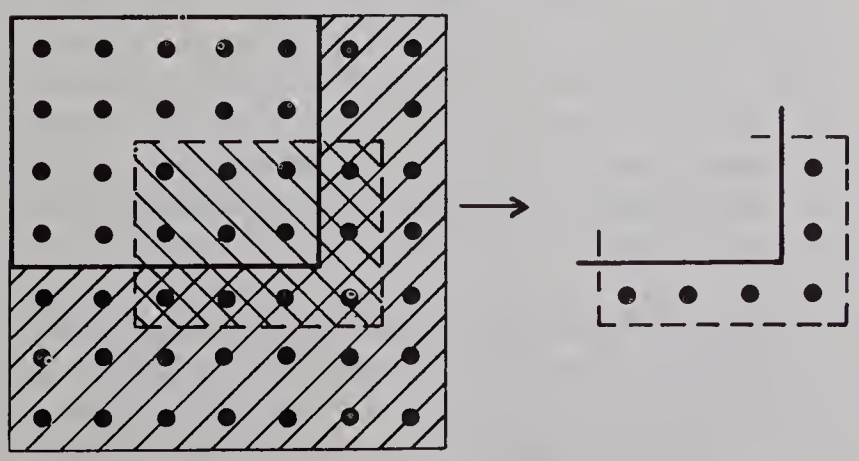
A player selects two cards from the first group and one from the second group. He then places one of the two on each of the "and" or the "or" card. It might look like this:



Now consider the board. How many dots could answer "yes" to both questions?—both (1) and (2):

(1) Is it true you *are not* inside the red line?

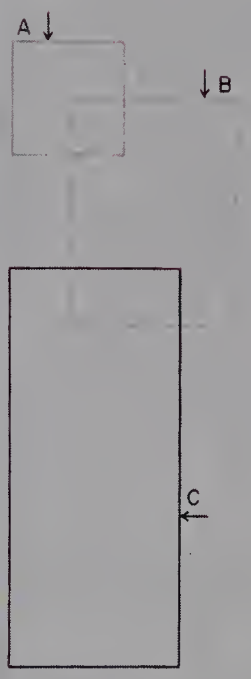
(2) Is it true that you *are* inside the green line?



Only 6 dots could answer "yes" to both of the questions—and that is the player's score: 6.

The scale for this sketch is:

Area indications	Area		Perimeter	
a. In G (green)	6	sq. ft.	11	ft.
b. In B (blue)	$11\frac{1}{2}$	sq. ft.	15	ft.
c. In B and G	$3\frac{3}{4}$	sq. ft.	8	ft.
d. In B or G	$13\frac{3}{4}$	sq. ft.	18	ft.
e. In B but not in G	$7\frac{3}{4}$	sq. ft.	17	ft.
f. In G but not in B	$2\frac{1}{4}$	sq. ft.	9	ft.



The scale for this sketch is:

Area indications	Area		Perimeter	
g. In A (green)	100	sq. mi.	40	mi.
h. In B (blue)	300	sq. mi.	70	mi.
i. In C (black)	525	sq. mi.	100	mi.
j. In A and B	25	sq. mi.	20	mi.
k. In B and C	50	sq. mi.	30	mi.
l. In A or B	375	sq. mi.	90	mi.
m. In B or C	775	sq. mi.	140	mi.
n. In A or C	625	sq. mi.	140	mi.
o. Not in B	550	sq. mi.	140	mi.
p. Not in C	325	sq. mi.	90	mi.
q. Not in A	750	sq. mi.	140	mi.
r. Not in B or C	75	sq. mi.	40	mi.
s. Not in A or B	475	sq. mi.	100	mi.
t. Not in A or C	225	sq. mi.	70	mi.

Had the center card been "or" instead of "and," then the question is: How many dots could answer "Yes" to either question (1) or question (2)? Then the score would be 35—(49 − 14 = 35)—all the dots in any part of the area shaded in the sketch above.

Players take turns and each accumulates a score. The first player to reach a predetermined total score "wins" the game. There are 72 possible combinations that can be summarized, as follows:

R, G, B stand for In red, In green, In blue. R, G, B stand for Not in red, Not in green, Not in blue.

	R	G	B	R	G	B
R	20	26	26	49	42	43
G	26	12	17	37	49	44
B	26	17	9	32	41	49
R	49	37	32	29	43	43
G	42	49	41	43	37	45
B	43	44	49	43	45	40

	R	G	B	R	G	B
R	20	6	3	0	14	17
G	6	12	4	6	0	8
B	3	4	9	6	5	0
R	0	6	6	29	23	23
G	14	0	5	23	37	32
B	17	8	0	23	32	40

70

sum	5	16	60	19	13	15	12	7	10	17	16	
oddends →	2	9	10	7	8	4	5	4	7	2	13	← factors
oddends →	3	7	50	12	5	11	7	3	3	15	3	← factors
	6	63	500	84	40	44	35	12	21	30	39	product

sum + 10 (s + 10)	20	22	22	21	24	22	24	30	20	27	21	
sum (s)	10	12	12	11	14	12	14	20	10	17	11	
oddends →	4	9	6	7	8	5	7	10	5	8	5	← factors
oddends →	6	3	6	4	6	7	7	10	5	9	6	← factors
	24	27	36	28	48	35	49	100	25	72	30	product (p)
	31	34	43	35	55	42	56	107	32	79	37	product + 7 (p + 7)

x+y+9	15	20	17	$10\frac{1}{2}$	10	$12\frac{3}{4}$	10	$11\frac{1}{2}$	14	$17\frac{1}{2}$	22	
2·(x+y)	12	22	16	3	2	$7\frac{1}{2}$	2	5	10	17	26	
x+y	6	11	8	$1\frac{1}{2}$	1	$3\frac{3}{4}$	1	$2\frac{1}{2}$	5	$8\frac{1}{2}$	13	
x	2	5	4	1	$\frac{1}{2}$	$\frac{3}{4}$	.5	$\frac{1}{2}$	2	7	6	x
y	4	6	4	$\frac{1}{2}$	$\frac{1}{2}$	3	.5	2	3	$1\frac{1}{2}$	7	y
	8	30	16	$\frac{1}{2}$	$\frac{1}{4}$	$2\frac{1}{4}$	.25	1	6	$10\frac{1}{2}$	42	x·y or xy
	23	45	31	$15\frac{1}{2}$	$15\frac{1}{4}$	$17\frac{1}{4}$	15.25	16	21	$25\frac{1}{2}$	57	xy+15
	16	60	32	1	$\frac{1}{2}$	$4\frac{1}{2}$	.5	2	12	21	84	2·xy or 2xy

m+n	.5	1.0	1.5	5.8	10.2	.6	18	1	1.5	9.9	2.4	5
m	.2	.4	.5	5	1.2	.3	9	.5	.7	9	1.2	2.5
n	.3	.6	1	.8	9	.3	9	.5	.8	.9	1.2	2.5
mn	.06	.24	.5	4	10.8	.09	81	.25	.56	8.1	1.44	6.25

70

Children ought to play the game without these two charts so that each play must be thought out. After a very short time, the idea of "and" and "or" will become clear.

Notes to accompany *Work Text* page 70

Divide the class into groups of from two to four each. Ask each group to see whether or not they can find out for themselves how each of the four charts on this page ought to be completed.

A group "pattern" hunt

The first chart is quite familiar to most students, and would probably be handled without difficulty even if all the words were omitted. In the next two exercises, however, students are confronted with pattern names of increasing complexity. Avoid discussion of these "pattern indicators" as anything other than just that. Is this algebra? Well, it is a sort of puzzle in which pattern indicators are clues—a mystery in which these clues can lead to solutions. Approached in this fashion, students must think their way through to reasonable answers—and this process is usually quite effective in small groups.

The question posed in the upper right-hand column of the last chart will be challenging—but solvable. It can be restated as a problem in an algebra textbook that would look like this:

$$2(x+y) = 26$$
$$xy + 15 = 57$$
$$x = \_\_\_\_$$
$$y = \_\_\_\_$$

But this form would require too much explanation at this point, and throws a spotlight on manipulations rather than on using pattern indicators as clues to help think through to a solution of a puzzle.

The final chart uses an alternate means of pointing to the same patterns employed in the first example.

Notes to accompany *Work Text* pages 71-73

One of the exciting adventure stories in the history of measurement is outlined briefly on *Work Text* page 71: the decision to adopt the meter as a standard unit of measure. A wealth of information on this subject can be found in the *20th Yearbook of the National Council of Teachers of Mathematics* (Bureau of Publications, Teachers College, Columbia University, New York, 1948) entitled, "The Metric System of Weights and Measures." This volume is written in language with which your students should have no difficulty.

An enduring standard

The story is that of the efforts of modern man to find a unit of measure based on some enduring standard —not a foot, or hand, or cubit, or pace—but "one ten-millionth of the arc of the great circle from the North Pole through Dunkirk, France and Barcelona, Spain, and on to the equator."

Men of science and mathematics in the eighteenth century felt sure that distances would never change and that at least 1/10th of it could be surveyed on land. They did not anticipate that the actual contour of the earth was not spherical, but somewhat flattened near the North and South Poles, nor that minute changes in that contour might occur over the centuries.

The most important aspect of the story is that each step in it points to the fact that measurement deals always with approximations. The first surveyors reached a certain conclusion about the length of a bar to represent the meter. More exact instruments were devised—and a closer approximation was made. Always the same development—finer

instruments and more careful observations lead to closer "approximation," but the results await even more accurate instruments for a still closer estimate.

Then, after agreement was reached on a certain "bar" the standard, it needed to be kept at a constant temperature and in the same place. Other countries needed copies of the bar. So, many approximate copies were made. From this group, those were selected which seemed closest to the standard in length. Were they "exactly" the same length? No, man cannot make perfect copies of any object.

Modern techniques of measurement can achieve results of amazing accuracy—but there is always a postscript, such as "correct to the 7th decimal point."

Understanding man's quest as one for "closer approximations" and not for "perfect accuracy," helps children understand measurement. When they are asked to measure an object, we hope they will begin by asking: "To what degree of accuracy do you want the measurements made?" Measurements are always approximate!

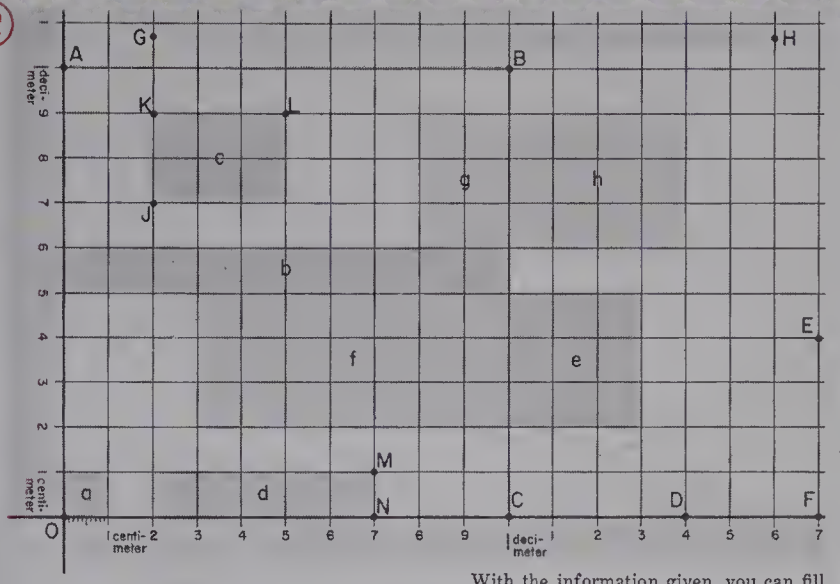
But when working with sketches such as those on *Work Text* pages 72 and 73, there is a supposition implied: "If this sketch were an accurate representation, then . . ." So, there are actually two kinds of questions: Consider the length of the line segment $\overline{CF}$. It is apparent that it is to be considered 7 centimeters long. The question about the length of line segment $\overline{CE}$ is another matter at this point. If we can only measure it, the result is going to be approximate. So, we are really asking: "If this sketch were an accurate representation of a square grid with lines 1 centimeter apart, then how long is the line segment $\overline{CE}$ to the *nearest centimeter* (or half-centimeter, or nearest millimeter, etc.)?

Later on, we will ask for the length of line segment $\overline{CE}$ "without measurement" and expect as an answer, to the nearest hundredth:

$$\sqrt{7^2 + 4^2} = \sqrt{49 + 16} = \sqrt{65} = 8.06$$

However, this approximation is not one of "measurement" but rather of "computation," and, in computation, the degree of accuracy has nothing to do with measurements and can be made as close as desired; $\sqrt{65}$ is not a rational number, but can be expressed as a never-ending decimal, correct to as many places as you want to compute it.

After completing page 73, consider the differences between the charts in the left-hand column and those in the right-hand column—between computa-



This diagram is full size. The line segment between points A and B is one decimeter long (one-tenth of a meter).

The line between points M and N is one centimeter long. The line between A and B is 10 centimeters long.

The distance between each small mark below the green square marked "a" is one millimeter — one-tenth of a centimeter.

Area

The small square marked "a" is one centimeter by one centimeter, and is called "one square centimeter."

The square whose corners are marked "A, B, C and O" is one decimeter by one decimeter, and the area inside the square is called "one square decimeter."

So, the area marked "b" is ten square centimeters, or one-tenth of a square decimeter.

With the information given, you can fill in the chart below, expressing the length of each line segment in meters.

	meters	decimeters	centimeters	millimeters
M N	.01	.1	1	10
D F	.03	.3	3	30
A B	.1	1 or 1.0	10	100
G H	.14	1.4	14	140
D E	.05	.5	5	50
C E	.08	.8	8(.)	80
C F	.07	.7	7	70
B C	.1	1.0	10	100
J K	.02	.2	2	20
N D	.07	.7	7	70
N F	.1	1.0	10	100

73 Small green block "a" is one square centimeter.

The square with corners marked "A, B, C and O" is one square decimeter.

How many square centimeters are there in a square decimeter? 100

How many square decimeters are there in a square meter? 100

How many square millimeters are there in a square centimeter? 100

To summarize:

With this information, you can fill in the chart below.

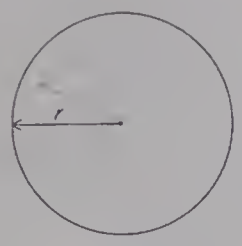
	square meters	square decimeters	square centimeters	square millimeters
a	.0001	.01	1	100
c	.0006	.06	6	600
d	.0005	.05	5	500
e	.0009	.09	9	900
b	.001	.1	10	1,000
f	.0021	.21	21	2,100
h	.0012	.12	12	1,200
(in square) A,B,C,O	.01	1	100	10,000
a + d	.0006	.06	6	600
g	.0006	.06	6	600
g + h	.0018	.18	18	1,800
f + e	.003	.3	30	3,000
f + b	.0031	.31	31	3,100
(in triangle) E,D,F,E	.0006	.06	6	600
(in triangle) E,C,F,E	.0014	.14	14	1,400
(in triangle) E,C,D,E	.0008	.08	8	800

	yards	feet	inches	16th of an inch
M N	$\frac{1}{36}$	$\frac{1}{12}$	1	16
D F	$\frac{1}{12}$	$\frac{1}{4}$	3	48
A B	$\frac{5}{18}$	$\frac{5}{6}$	10	160
G H	$\frac{7}{18}$	$1\frac{1}{6}$	14	224
C E	$\frac{2}{9}$	$\frac{2}{3}$	8	128
C F	$\frac{7}{36}$	$\frac{7}{12}$	7	112
J K	$\frac{1}{18}$	$\frac{1}{6}$	2	32

	sq. yards	sq. feet	sq. inches	sq. 16th of an inch
a	$\frac{1}{1,296}$	$\frac{1}{144}$	1	256
b	$\frac{5}{648}$	$\frac{5}{72}$	10	2,560
c	$\frac{1}{216}$	$\frac{1}{24}$	6	1,536
e	$\frac{1}{144}$	$\frac{1}{16}$	9	2,304
f	$\frac{7}{432}$	$\frac{7}{48}$	21	5,376

Now you have an idea of why, in 1893, the U.S., by Act of Congress, adopted the meter as our official standard of measure.

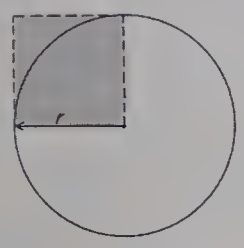
Why do you think we still use yards and feet?



A line drawn from the center to any point on a circle's perimeter is called a "radius;" a radius of the circle above is labeled "r."

If a circle had a shorter radius – such as the length of the green arrow – would its area be larger or smaller?

If its radius were longer, than r suggested by the blue arrow – would the area of the circle be larger or smaller than the one in the sketch above with radius r?



How many tiles the size of r^2 ($r \cdot r$ or "r squared") would be needed to cover the circle and the area inside the circle?

Could you place two whole tiles with size r^2 inside the circle so they did not overlap and no point on the tile would be outside the circle? No

How many such tiles would you need to be sure to cover all points on the circle and all inside the circle? Four or more

So, we see that the area of the circle is more than r^2 and less than $4r^2$.

That's not a very close approximation.

Suppose we use tiles that are $\frac{1}{2}r$ by $\frac{1}{2}r$ — or tiles with an area of $\frac{1}{4}r^2$?

Notice Diagram I on the next page. This is one way this might be done. The tiles inside the green line are clearly "inside" the circle. The tiles inside the blue line clearly cover all points on and inside the circle.

Put a dot in every tile that touches any point on the circle. Complete the blue line so it includes all tiles that touch the circle at any point.

How many tiles are there inside the blue line in Diagram I? 24

Now we know that the area is somewhere between 4 of these smaller tiles and 24 of them.

This is a very poor approximation, but it is only the beginning.

Now, suppose we use still smaller tiles. Let's make each tile $\frac{1}{4}r$ by $\frac{1}{4}r$ — or $\frac{1}{16}r^2$.

Follow the same procedure as before in Diagram II.

How many tiles are inside the green line — tiles that do not touch any point on the circle? 32

How many tiles are used to cover all the points on and inside the circle? 68

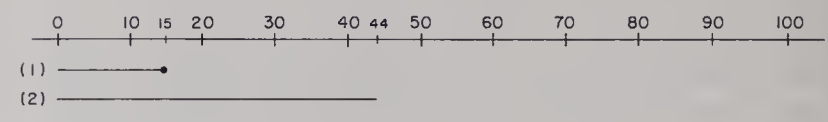
Now we know that the area is somewhere between 32 of these still smaller tiles and 68 of them.

32 $2r^2$

68 $4\frac{1}{4}r^2$

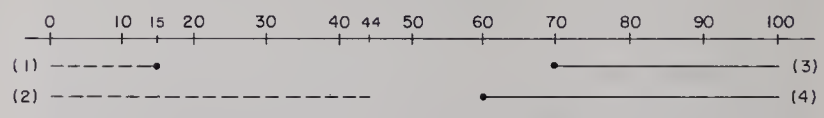
"I'm thinking of a number between 0 and 100. You can ask questions in the form of, 'Is it more than . . . ' or 'Is it less than . . . ', and I will answer yes or no. We will keep track of your questions and my answers on a number line."

1) "Is it more than 15?" Yes.
2) "Is it less than 44?" No.

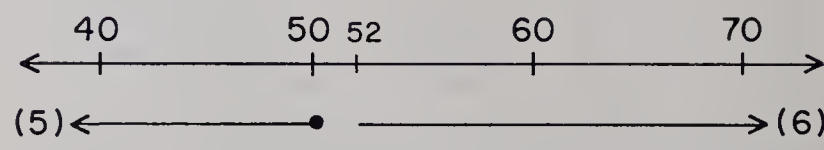


The first question ruled out 15 and all numbers less than 15—all numbers we would use to label points on the number line above the line labeled (1). The second question ruled out all the points above the line labeled (2).

3) "Is it less than 70?" Yes . . . rules out all numbers 70 and above.
4) "Is it more than 60?" Yes . . . rules out 60 and all numbers less than 60.

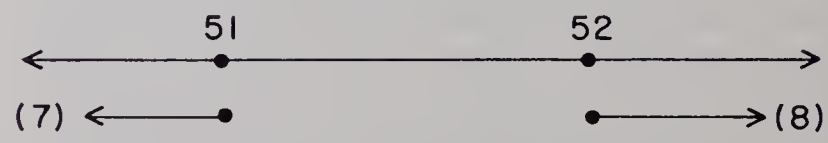


5) "Is it more than 50?" Yes.
6) "Is it more than 52?" No.



(Note that since question (5) eliminates 50 and all numbers less than 50, a point is indicated under 50; it, too, is excluded. But question (6) eliminates numbers greater than 52—but not 50 itself. So, the line indicating the elimination does not show a point under 52.)

7) "Is it more than 51?" Yes. . .eliminates 51 and all smaller numbers.
8) "Is it less than 52?" Yes. . .eliminates 52 and all larger numbers.



It must be between 51 and 52.

9) "Is it more than 51.5?" Yes.
10) "Is it more than 51.7?" Yes.
11) "Is it more than 51.9?" No.

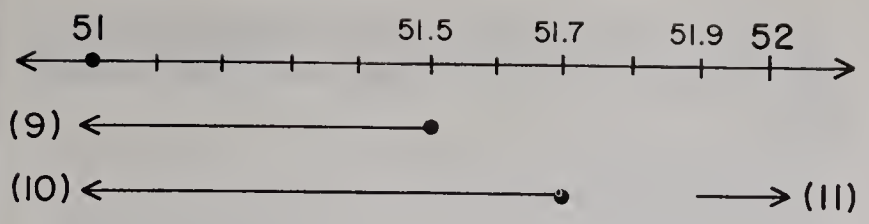
tion of results in the "metric system" and the "English system" of linear measurement.

Have some children investigate the English system of money and compare it with the money systems of France, the U.S., or Canada. The metric system (or any system that parallels the decimal system of numeration) is always a more manageable system.

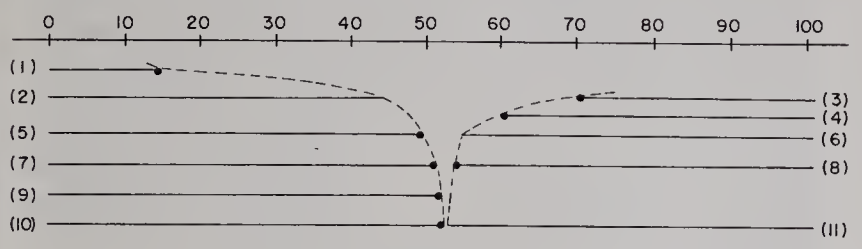
Finally, why doesn't the United States adopt the metric system of weights and measures? Students can talk this over with their parents, and hosts of unexpected reasons will turn up. For example, if we did, all screws and nuts and bolts would have to be changed, and all the machines that make the screws and nuts and bolts, etc.

Notes to accompany *Work Text* pages 74-77

While the next three pages are a search for the relationship between the area of a circle and the square of its radius—$A = \pi r^2$—we are particularly interested in the method. Before considering *Work Text* pages 75 through 78, play this game a few times.



My number is 5.8249176805 (and I'll keep writing decimals as you get closer). The questions thus far have produced this result:



The questions begin to squeeze in on my number from both sides; they are trapping it in a smaller and smaller gap. The game can be given an ending by an arbitrary rule: As soon as my number is trapped in a gap not more than .3 wide, the game is done – and I collect a point for every question you've asked. In the example above, the gap was narrowed to .2 by 11 questions, so I get 11 points. It's your turn next, and this, of course, is simply another version of "Fewest Questions."

Trapping the number

Play several games similar to the one above, so that this concept of trapping a number between two other numbers is clearly fixed. This is precisely what we are going to do in pages 75 through 78: trap a number that represents the relationship between the area of a circle and square of its radius (and again on *Work Text* pages 81, 82 and 83).

We are looking for a number to complete the statement:

$$\_\_\_\_ \times (\text{radius} \times \text{radius}) = \text{area, or } \_\_\_\_ r^2 = A$$

We will start with a number that is obviously too small and another which is obviously too large. Thus, we have now caught our number between a number for an outer area and a number for an inner area.

Diagram I on *Work Text* page 75 moves to this end. Instead of using tiles the size of the radius squared, we use tiles one-fourth as large. In the diagram we notice that four of these tiles (in green) are clearly inadequate to cover the area inside the circle, and $4 \times 1/4 = 1$; so, the area is more than $1 \times r^2$. This is clearly too small, because none of the four green tiles even touches the circle.

We also find that 24 of the tiles more than cover an area the size of the circle – all points of the circle are well inside the outline of the 24 tiles. Since $24 \times 1/4 = 6$, we know that the area is certainly not as large as $6 \times r^2$.

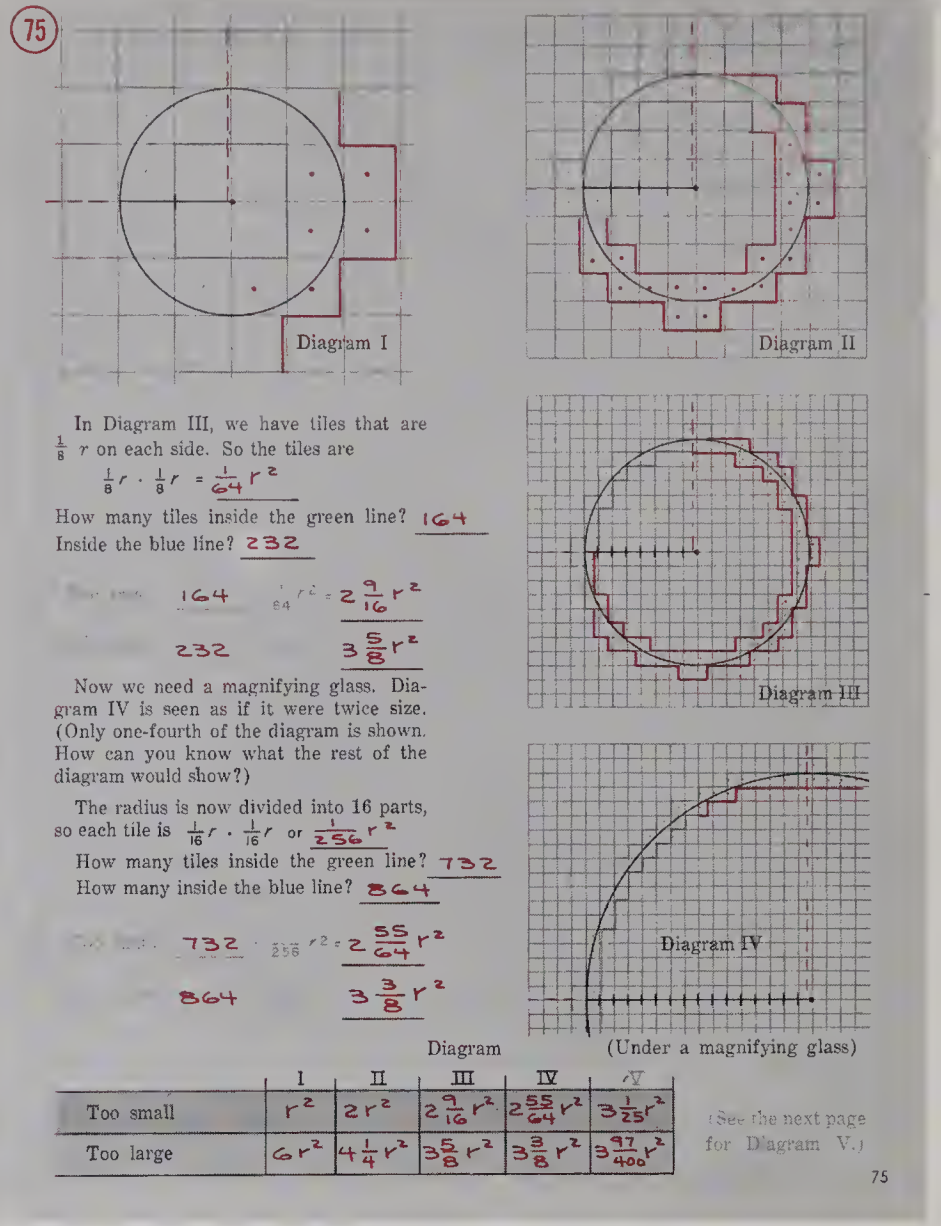
So, we have caught the area between $1 \times r^2$ and $6 \times r^2$. We keep using smaller and smaller tiles: 1/16 of r^2, 1/64 or r^2, and (under a magnifying glass) 1/256 or r^2 . . . and, finally, 1/1600 of the square of the radius.

On a series of number lines on *Work Text* page 77, we record this development in a way similar to the way we kept score in "Fewest Questions." The elusive number is successfully caught between:

(a) 1 and 6	(Diagram I)
(b) 2 and 4 1/4	(Diagram II)
(c) 2 9/16 and 3 5/8	(Diagram III)
(d) 2 55/64 and 3 3/8	(Diagram IV)
(e) 3 1/25 and 3 97/400	(Diagram V)

Or, it is trapped between 3.04 and 3.24 (the average of which is 3.14 and, our number correct to four decimal places is 3.1416; so, we come very close indeed.

The particular scheme we use to arrange the tile or their size is immaterial. We could have made


75
Diagram I
Diagram II
In Diagram III, we have tiles that are 1/8 r on each side. So the tiles are
1/8 r · 1/8 r = 1/64 r²
How many tiles inside the green line? 164
Inside the blue line? 232
164 1/64 r² = 2 9/16 r²
232 3 5/8 r²
Diagram III
Now we need a magnifying glass. Diagram IV is seen as if it were twice size. (Only one-fourth of the diagram is shown. How can you know what the rest of the diagram would show?)
The radius is now divided into 16 parts, so each tile is 1/16 r · 1/16 r or 1/256 r²
How many tiles inside the green line? 732
How many inside the blue line? 864
732 1/256 r² = 2 55/64 r²
864 3 3/8 r²
Diagram IV
Diagram (Under a magnifying glass)
I II III IV V
Too small r² 2r² 2 9/16 r² 2 55/64 r² 3 1/25 r²
Too large 6r² 4 1/4 r² 3 5/8 r² 3 3/8 r² 3 97/400 r²
(See the next page for Diagram V.)
75

76

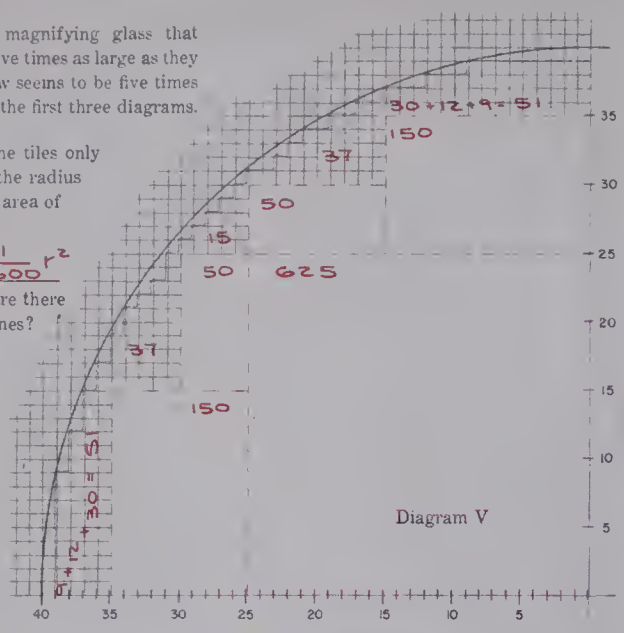
Next we use a magnifying glass that makes things look five times as large as they are. Our radius now seems to be five times as long as it was in the first three diagrams.

We have made the tiles only $\frac{1}{40}$ of the length of the radius on each side, so the area of each tile is

$\frac{1}{40}r \cdot \frac{1}{40}r = \frac{1}{1600}r^2$

How many tiles are there inside the green lines? (You can find the number in the large blocks without drawing them.) There are 1,216 in this diagram of $\frac{1}{4}$ of a circle. In a diagram of the whole circle, there would be

4,864



How many more than that are there inside the blue line in this fourth of the total diagram? 81 In a similar diagram showing the whole circle, there would be four times as many, or a total of 324 blue than green. So, the total number inside the blue line would be 5,188

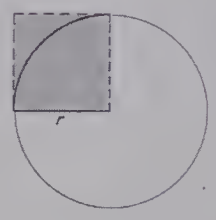
How many of these small tiles would be required to cover a square using the radius in Diagram V as the length of each side?

1,600

Too small $4{,}864 \cdot \frac{1}{1600}r^2 = 3\frac{1}{25}r^2$

5,188 $3\frac{97}{400}$

(Record these results in the chart on the previous page.)



The green block above suggests our idea of r^2. If we cut four of these blocks into 1,600 tiny squares each, we know we could put at least 4,864 of the tiny squares completely inside our circle; and, we would not need more than 5,188 to cover every point on and inside the circle.

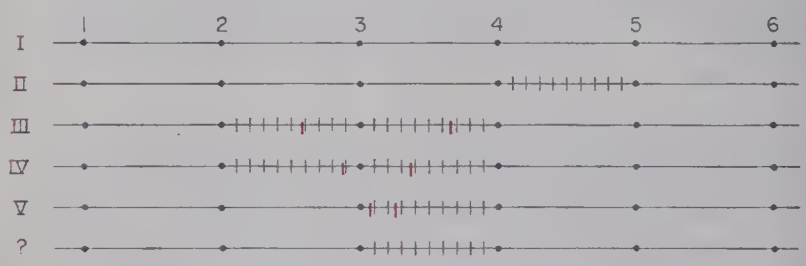
So, the area of our circle is somewhere in between.

77

Express the results of the five studies in decimals and record below . . . (to the nearest hundredth)

	I	II	III	IV	V	?
Too small	$1.0r^2$	$2.0r^2$	$2.56r^2$	$2.86r^2$	$3.04r^2$	
Too large	$6.0r^2$	$4.25r^2$	$3.62r^2$	$3.37r^2$	$3.24r^2$	

What do you notice? Now, indicate these results on a series of number lines.



We could keep magnifying our diagrams and divide the radius into the smallest units we could handle.

In each experiment, the difference between the area that is clearly "too small" and the area that is "too large" would become less and less.

Could we design an experiment with such a large radius and such small divisions that we would reduce that difference to zero?

No

As we kept recording our continuously magnified experiments on a number line, would they crowd in from both directions on a certain point? Yes

We know that such a point must be located somewhere between the points labeled 3.04 and 3.24. What point is halfway between? 3.14

Our experiments have led us to believe that the point 3.14 is very close to the number we're looking for.

By many similar experiments and by use of higher mathematics, we know that the point we are crowding is hard to name. It cannot be written as a fraction, using any numbers we know as numerator and denominator. We can name a point that is as close as we want — but that is all. We can write a decimal fraction as long as we like, but it never comes to a neat conclusion. The decimal fraction starts out

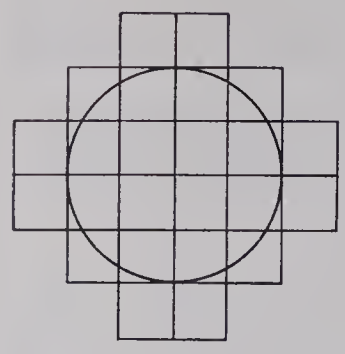
3.

and those dots go on and on and on . . . .

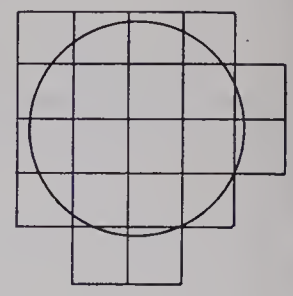
For most calculations, that is close enough. Often we use $3\frac{1}{7}$ or $\frac{22}{7}$ as an approximation. What decimal fraction does that approximation give?

3.1428571.....

our first approximation closer by rearranging the tile. By moving the circle down and right, we could have:



Between $4 \times 1/4$ and $24 \times 1/4$, or Between 1 and 6



Between $4 \times 1/4$ and $20 \times 1/4$, or Between 1 and 5

If we find the average of each of these examples:

$1 + 6 \quad 7 \quad 3\ 1/2$

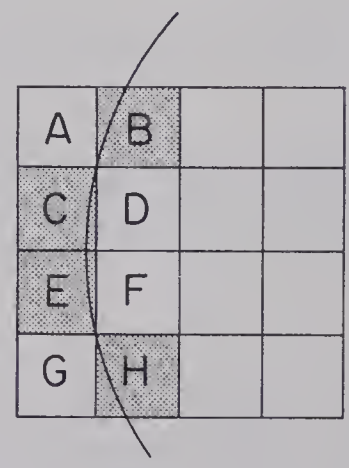
$$\frac{1+6}{2} = \frac{7}{2} = 3\ 1/2$$

$$\frac{1+5}{2} = \frac{6}{3} = 3$$

$$\frac{3\ 1/2 + 3}{2} = \frac{6.5}{2} = 3.25$$

which is, itself, a reasonably close approximation of 3.1416.

Different groups within the class can make their own studies of this relationship between the square on the radius and the area of a circle. If they see a case in which they cannot tell whether the circle touches a tile or not, such as this—



always assume it touches. This will widen the gap between the numbers for the inner area and the outer area—between the number that is obviously too small and the number that is obviously too large. We will return to this idea often; when numbers are elusive, we will keep trapping them between in a smaller and smaller gap.

Notes to accompany *Work Text* pages 78-79

The number we have trapped within reasonable close limits—between 3.04 and 3.14—has been studied for centuries. The upper and lower bound between which it is trapped have come very close together. We know it can never be located exactly by a decimal fraction, but it can be closely approximated, as in the heading on *Work Text* page 78, correct to the 50th decimal place. (It is, of course, an irrational number: a number that cannot be written in the form of "a/b" in which a and b are both whole numbers.)

On *Work Text* pages 78 and 79, we put approximations of π (pronounced "pie") to work. We know that it would take about 3 1/7 or 3.1416 tiles the size of the radius squared to cover a circle. In each example, draw a sketch similar to that at the top of page 78, noting length of the radius and the size of the square which has the length of a radius as the length of its sides. Finally, direct attention toward finding examples in life in which one needs to know the area of a circle in terms of the radius. Also, examples in which the area is known and the radius must be determined.

For instance: A farmer wants to build a circular bin that has a floor space of more than 310 feet and less than 320 feet. Find a diameter (twice the radius) that would meet these specifications. (10 ft. radius: $10 \times 10 \times 3.1416 = 314.16$ and 314.16 square feet meets the condition.)

How much will a round steel plate weigh if it is 2 feet across and weighs 10 pounds per square foot? (Radius is 1 sq. ft.; area is 3.1416 square feet and 3.1416 times 10 pounds which is about 31.416 pounds.)

Then present "headlines" and have the students suggest appropriate stories:

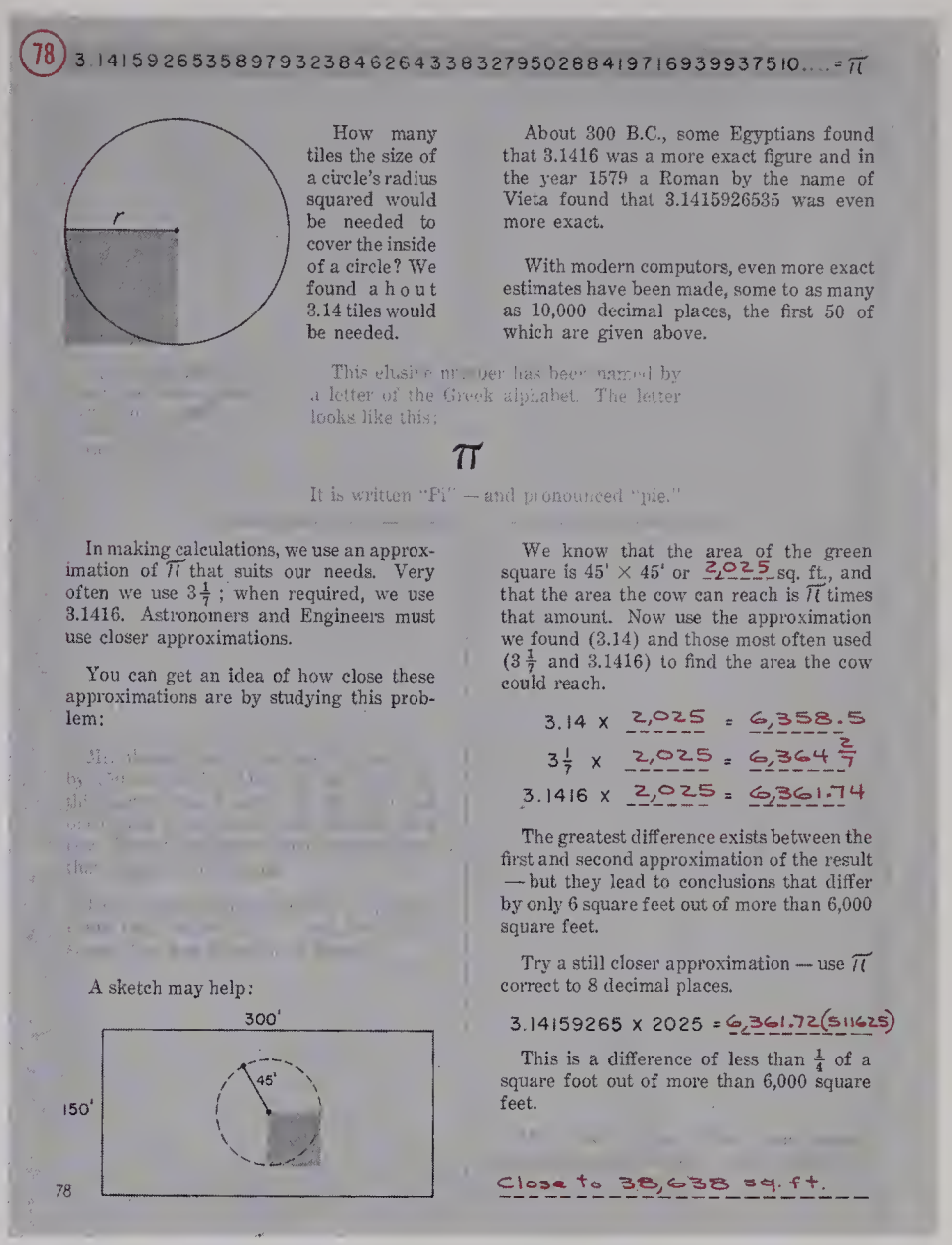
$$3\ 1/7 \times 81 = \underline{254\ 4/7}$$

"If a rope is 9 feet long and tied to a post, a cow has about 254 1/2 square feet of grazing area."

"A circular flower garden with a diameter of 18 feet would have an area of more than 250 square feet."

$$3.1416 \times \_\_\_\_\_ = 31{,}416$$

"A circular plot of ground to include about 30,000 square feet must have a radius of about 100 feet, because $3.1416 \times 100^2 = 31{,}416$."



78 3.14159265358979323846264338327950288419716939937510....= π

How many tiles the size of a circle's radius squared would be needed to cover the inside of a circle? We found about 3.14 tiles would be needed.

About 300 B.C., some Egyptians found that 3.1416 was a more exact figure and in the year 1579 a Roman by the name of Vieta found that 3.1415926535 was even more exact.

With modern computors, even more exact estimates have been made, some to as many as 10,000 decimal places, the first 50 of which are given above.

This elusive number has been named by a letter of the Greek alphabet. The letter looks like this:

π

It is written "Pi" — and pronounced "pie."

In making calculations, we use an approximation of π that suits our needs. Very often we use $3\frac{1}{7}$; when required, we use 3.1416. Astronomers and Engineers must use closer approximations.

You can get an idea of how close these approximations are by studying this problem:

A sketch may help:

We know that the area of the green square is 45' × 45' or 2,025 sq. ft., and that the area the cow can reach is π times that amount. Now use the approximation we found (3.14) and those most often used ($3\frac{1}{7}$ and 3.1416) to find the area the cow could reach.

3.14 x 2,025 = 6,358.5

$3\frac{1}{7}$ x 2,025 = $6,364\frac{2}{7}$

3.1416 x 2,025 = 6,361.74

The greatest difference exists between the first and second approximation of the result — but they lead to conclusions that differ by only 6 square feet out of more than 6,000 square feet.

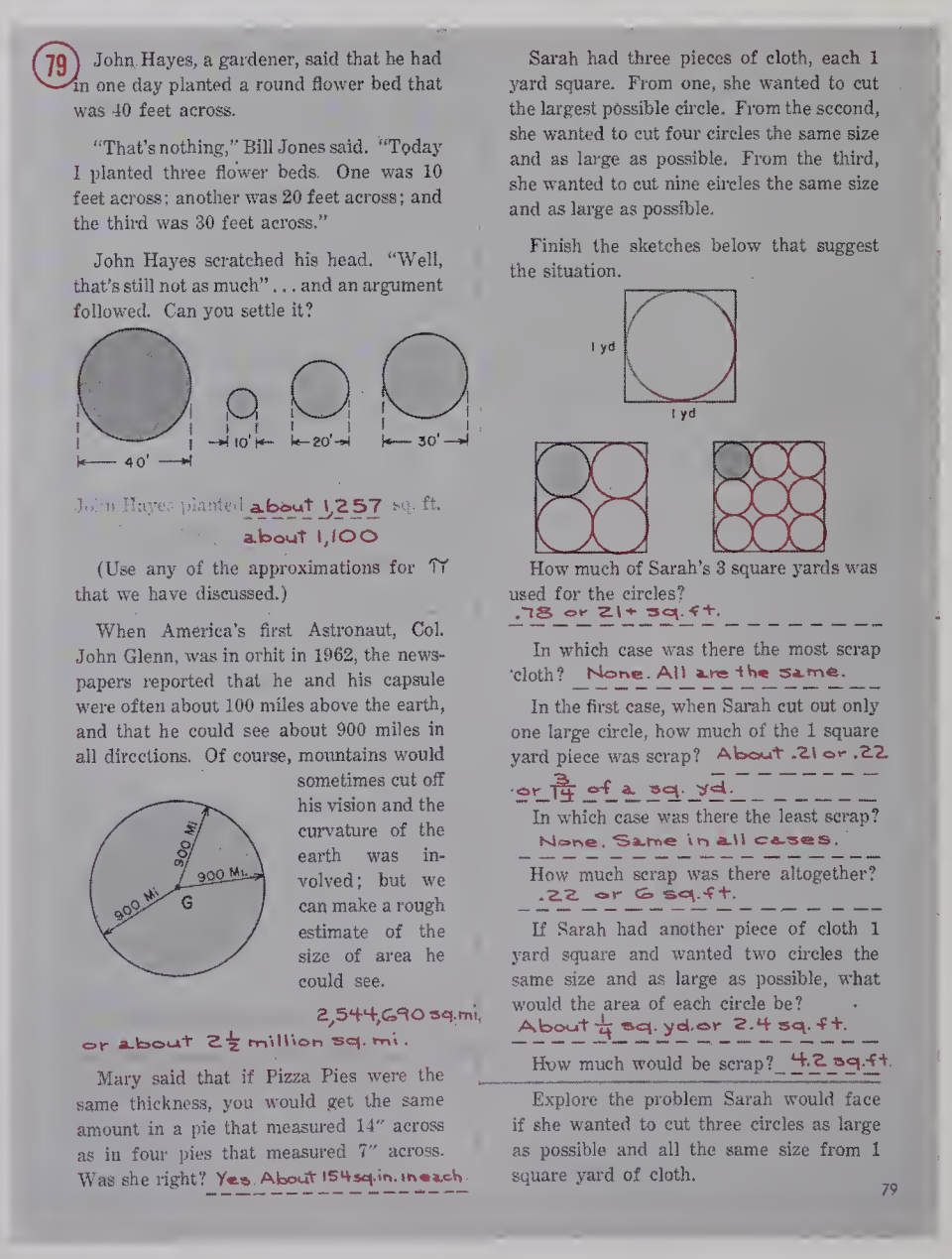
Try a still closer approximation — use π correct to 8 decimal places.

3.14159265 x 2025 = 6,361.72(511625)

This is a difference of less than $\frac{1}{4}$ of a square foot out of more than 6,000 square feet.

Close to 38,638 sq. ft.

78



79 John Hayes, a gardener, said that he had in one day planted a round flower bed that was 40 feet across.

"That's nothing," Bill Jones said. "Today I planted three flower beds. One was 10 feet across; another was 20 feet across; and the third was 30 feet across."

John Hayes scratched his head. "Well, that's still not as much"... and an argument followed. Can you settle it?

John Hayes planted about 1,257 sq. ft.
about 1,100

(Use any of the approximations for π that we have discussed.)

When America's first Astronaut, Col. John Glenn, was in orbit in 1962, the newspapers reported that he and his capsule were often about 100 miles above the earth, and that he could see about 900 miles in all directions. Of course, mountains would sometimes cut off his vision and the curvature of the earth was involved; but we can make a rough estimate of the size of area he could see.

2,544,690 sq.mi.
or about 2½ million sq. mi.

Mary said that if Pizza Pies were the same thickness, you would get the same amount in a pie that measured 14" across as in four pies that measured 7" across. Was she right? Yes. About 154 sq.in. in each

Sarah had three pieces of cloth, each 1 yard square. From one, she wanted to cut the largest possible circle. From the second, she wanted to cut four circles the same size and as large as possible. From the third, she wanted to cut nine circles the same size and as large as possible.

Finish the sketches below that suggest the situation.

How much of Sarah's 3 square yards was used for the circles?
.78 or 21+ sq. ft.

In which case was there the most scrap cloth? None. All are the same.

In the first case, when Sarah cut out only one large circle, how much of the 1 square yard piece was scrap? About .21 or .22 or $\frac{3}{14}$ of a sq. yd.

In which case was there the least scrap?
None. Same in all cases.

How much scrap was there altogether?
.22 or 6 sq. ft.

If Sarah had another piece of cloth 1 yard square and wanted two circles the same size and as large as possible, what would the area of each circle be?
About $\frac{1}{4}$ sq. yd. or 2.4 sq. ft.

How much would be scrap? 4.2 sq. ft.

Explore the problem Sarah would face if she wanted to cut three circles as large as possible and all the same size from 1 square yard of cloth.

79

A

c	15	21	4	.5	5	$\frac{1}{4}$	.5	5	.1
$c+d$	75	40	29	1.2	5.07	1	1	105	10.1
$c \cdot d$ or cd	900	399	100	.35	.35	$\frac{3}{16}$	$\frac{1}{4}$	500	1
d	60	19	25	.7	.07	$\frac{3}{4}$	$\frac{1}{2}$	100	10

B

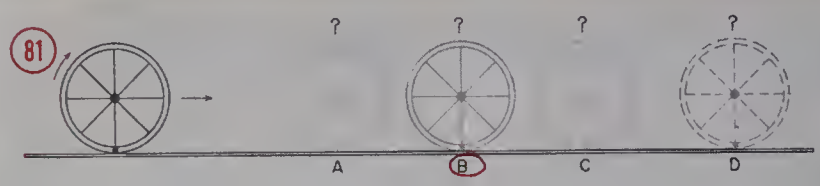
s	5	12	3	.7	$\frac{1}{2}$	$1\frac{1}{2}$	35	43	125
t	7	12	35	.9	$\frac{3}{4}$	$1\frac{1}{2}$	35	37	3
$s+t$	12	24	38	1.6	$1\frac{1}{4}$	3	70	80	128
st	35	144	105	.63	$\frac{3}{8}$	$2\frac{1}{4}$	1,225	1,591	375
$2 \cdot st$ or $2st$	70	288	210	1.26	$\frac{3}{4}$	$4\frac{1}{2}$	2,450	3,182	750
$2s+2t$	24	48	76	3.2	$2\frac{1}{2}$	6	140	160	256
$2s+t$	17	36	41	2.3	$1\frac{3}{4}$	$4\frac{1}{2}$	105	123	253

C

x	25	1.8	5	.3	$\frac{1}{4}$	11	9	.87	500
xy	125	1.26	5	.09	$\frac{1}{16}$	187	1.35	.087	2,500
y	5	.7	1	.3	$\frac{1}{4}$	17	.15	.1	5
$2x+y$	55	4.3	11	.9	$\frac{3}{4}$	39	18.15	1.84	1,005
$2x-y$	45	2.9	9	.3	$\frac{1}{4}$	5	17.85	1.64	995

D

$a \cdot b$ or ab	6.8	.49	1.44	2.56	13.5	2.10	1	10	.01
a	3.4	.7	1.2	8	1.5	.6	.1	.1	.1
b	2	.7	1.2	.32	9	3.5	10	100	.1
$a+b$	5.4	1.4	2.4	8.32	10.5	4.1	10.1	100.1	.2

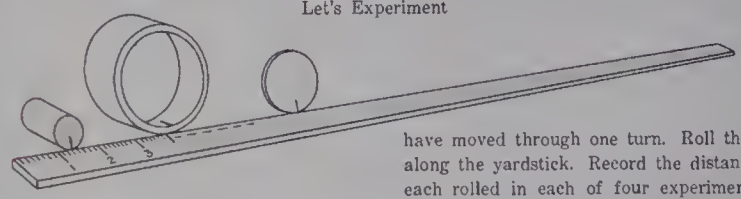


How far does a wheel roll in one turn?

Does a roller skate wheel go as far in one turn as a bicycle wheel? Which goes further in one turn—the large front wheel of a tricycle or its smaller back wheels?

In the diagram above, imagine that the black wheel rolls along to the right. It has a spot on it so we can tell when it has completed one turn. Which colored sketch indicates its position after one turn: A, B, C, or D?

Let's Experiment



You will need a yardstick and a pin.

Out of cardboard, cut wheels with diameters of 2", 4" and 8". Near their outside edges, put a mark so you can tell when they have moved through one turn. Roll them along the yardstick. Record the distances each rolled in each of four experiments. Add the results and divide by four to find "average distance."

Continue experimenting with other objects such as tin cans, rolls of tape, wheels—anything that is round.

Name of object	Diameter of object	Distance traveled in 1 turn 1st exp.	2nd.	3rd.	4th.	Average distance traveled	Average distance = ____	X Diameter
Cardboard wheel	2"						= ____	X ____
Cardboard wheel	4"						= ____	X ____
Cardboard wheel	8"						= ____	X ____
							= ____	X ____
							= ____	X ____
							____	X ____
							____	X ____
							____	X ____

Add the 8 numbers written above the blue box and divide by 8 to find their average.

Notes to accompany *Work Text* page 80

Refresh the students memory by repeating the "sum and product" pattern hunt:

Sum			14		8			
	2	3		7				
	8	9	4					
Product				77	15			

Then vary the form:

Sum				30	1		
Product			1/12	125	1/4		
	7	6	1/3				
	8	15					

Then let students in small groups, or independently, complete the puzzle-like examples on *Work Text* page 80.

Notes to accompany *Work Text* pages 81-83

"What is the distance around a circle?"

"If the length of the diameter of two circles is the same, are the distances around the circle the same?"

"If the length of the diameter of circle A is more than the length of the diameter of circle B, can you be sure which has the greater distance around?"

"Circumference is the distance around a circle."

"As the measure of the diameter increases, does the circumference increase?"

"Is there a definite relationship between the circumference and the length of the diameter of a circle?"

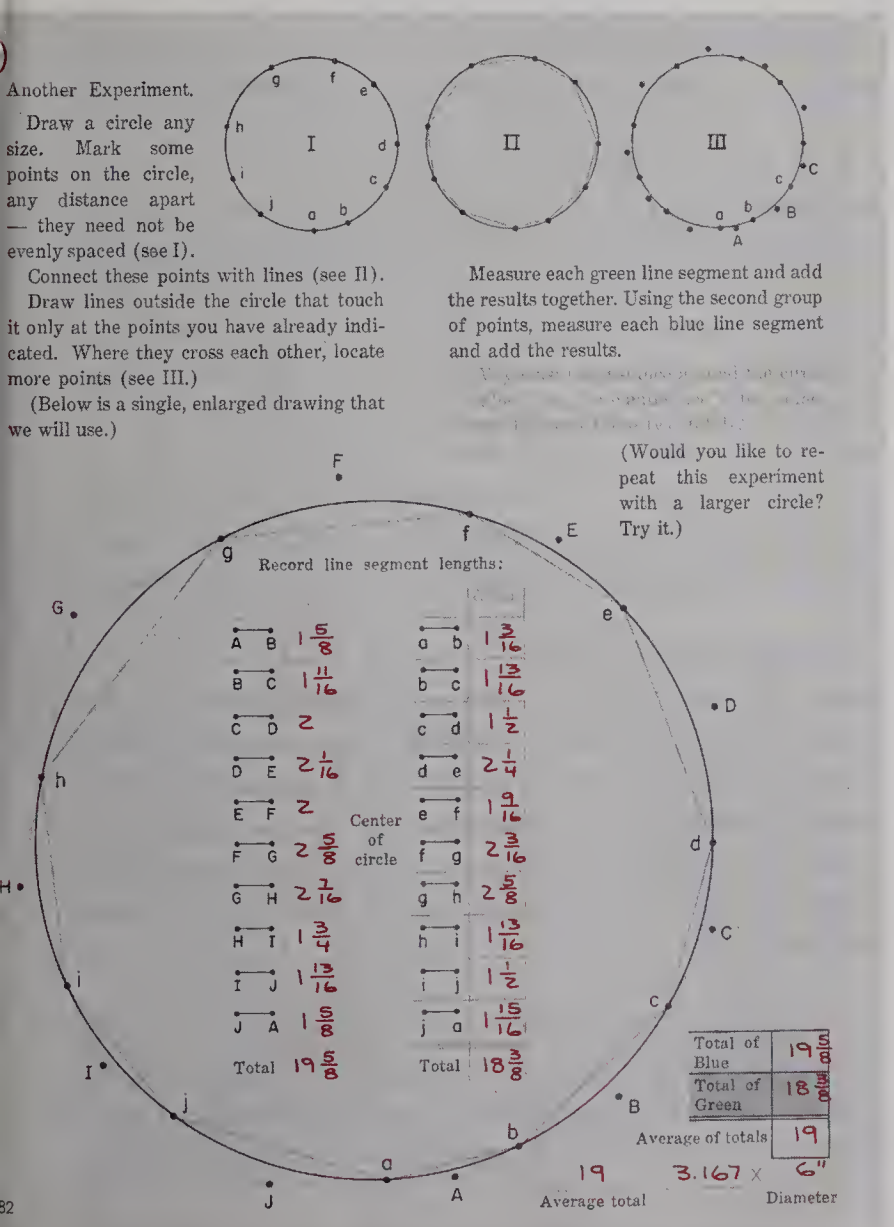
Let's experiment! The class can discuss several methods of determining the circumference. As shown on *Work Text* page 81, one way is to put a mark on the circle, and then roll the circle along a ruler until the mark has made one revolution. Another is to wrap a string around a circular object and then measure the amount of string required. It must be made clear that all experiments depend on human observations of results based on humanly constructed instruments of measurement—and are thus, at best, approximations. (This, of course, is true of all scientific experiments that involve measurements.)

In discussing methods of performing our experiments, consideration properly is given to the idea of finding an average of several measurements made according to several methods by several people. While we rely on this method as providing a reasonable result, we know full well that it might be that Deborah's first result, a single experiment, might be closer to the actual size than our average—but we have only limited reasons for having confidence in a single measurement.

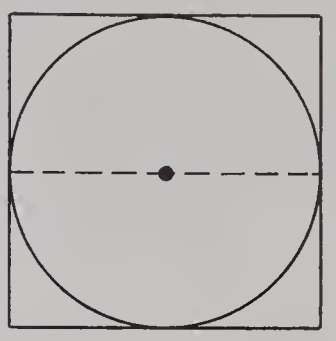
After collecting evidence about the circumference of a circle and its diameter, we find a number that completes the following as a true statement:

_____	=	?	×	_____
(average of measurement of circumference)				(average of measurement of diamter length)

When we have found this number to complete this statement about the results of experimentation with many objects, we note that the range between the largest and smallest measurement is very narrow. This suggests that there is probably a number that represents the relationship between the length of the diameter and the circumference of a circle. So, we find the average of these results, and accept it as a close approximation.


Another Experiment.
Draw a circle any size. Mark some points on the circle, any distance apart — they need not be evenly spaced (see I).
Connect these points with lines (see II).
Draw lines outside the circle that touch it only at the points you have already indicated. Where they cross each other, locate more points (see III.)
(Below is a single, enlarged drawing that we will use.)
Measure each green line segment and add the results together. Using the second group of points, measure each blue line segment and add the results.
(Would you like to repeat this experiment with a larger circle? Try it.)
Record line segment lengths:
Center of circle
Total 19 5/8
Total 18 3/8
Total of Blue 19 5/8
Total of Green 18 3/8
Average of totals 19
19 Average total
3.167 × 6" Diameter
82

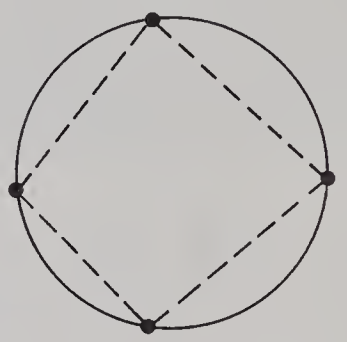
Work Text page 82 approaches our problem in a different way, yet still based upon experiment. Draw a circle on the board that is inscribed in a square:



"Suppose we measure the diameter and find that its length is 30 inches. What is the perimeter of the square?" Answer: 4 × 30 inches is 120 inches.

"What can you say about the circumference and be sure you are correct?" It is less than 120 inches.

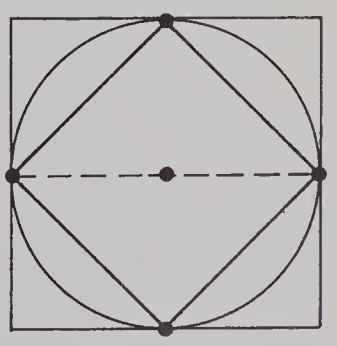
Add a square to the sketch—inscribed in the circle:



"Suppose you measure each side of the square that is inscribed in the circle and find it is close to 21 1/2 inches. What is an approximation of its perimeter?" Answer: 4 × 21 1/2 inches is 86 inches.

"What more can you now say about the circumference of the circle?" Answer: It clearly is more than 86 inches, as well as less than 120 inches.

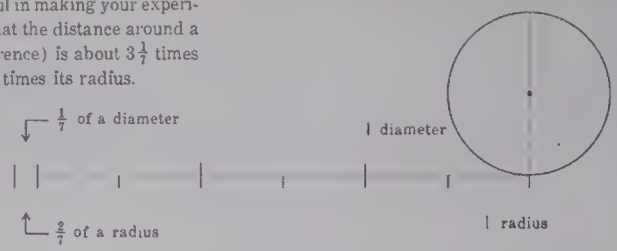
Consider another approach. On a circle, locate four points that are not equally spaced. Connect them with straight lines:



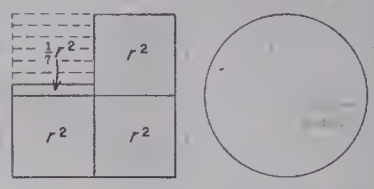
This forms what we call an "inner quadrilateral." If we measure each of its four sides, will its perimeter be less than the circumference? Yes! Now,

83

If you were careful in making your experiments, you found that the distance around a circle (its circumference) is about $3\frac{1}{7}$ times its diameter, or $6\frac{2}{7}$ times its radius.



That "about $3\frac{1}{7}$" sounds familiar. You found earlier that it takes about $3\frac{1}{7}$ squares, measuring the same on each side as the radius, to cover the area inside the circle.



You will remember that $3\frac{1}{7}$ is an approximation for a never-ending decimal:

3.1416 5926535... . = π (written "Pi")

Where we are speaking about circles, there is some shorthand you will meet very often.

is shorthand for "area"
is shorthand for "diameter"
is shorthand for "radius"
is shorthand for "circumference"
is a name for the number that is approximately $3\frac{1}{7}$ or 3.1416 etc.

To find the area of a circle, we multiply π times the radius times the radius. In shorthand:

To find the circumference of a circle, we multiply the diameter by π . In shorthand, this is:

Why can we find the circumference by multiplying the radius by $2 \cdot \pi$?

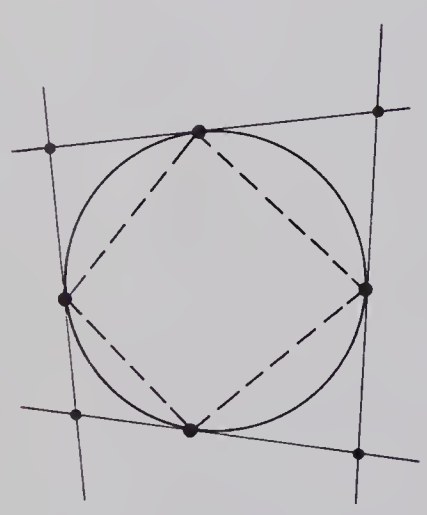
Length of Radius	Length of Diameter	Area of Circle	Circumference of Circle
$\frac{1}{2}$"	1"	$\frac{11}{14}$ sq. in.	$3\frac{1}{7}$
1"	2"	$3\frac{1}{7}$ sq. in.	$6\frac{2}{7}$
2"	4"	$12\frac{4}{7}$ sq. in.	$12\frac{4}{7}$
3"	6"	$28\frac{2}{7}$ sq. in.	$18\frac{6}{7}$
4"	8"	$50\frac{2}{7}$ sq. in.	$25\frac{1}{7}$
5"	10"	$78\frac{4}{7}$ sq. in.	$31\frac{3}{7}$
7"	14"	154 sq. in.	44
10"	20"	$314\frac{2}{7}$ sq. in.	$62\frac{6}{7}$

Is it true that you can find the area of the circles indicated above by finding $\frac{1}{2}$ of the radius times the circumference?

?

83

through each point draw a line that touches the circle only at that point (these lines are called "tangents").



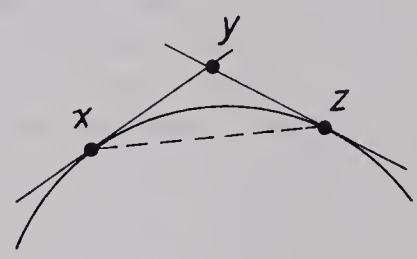
Note the points at which the tangents cross. Consider them as corners or vertices of what we call an "outer quadrilaterial." Can we be sure that measurements will give us an approximate perimeter that is more than the circumference?

"Can you say that the circumference is more than the perimeter of the inner quadrilateral and less than the perimeter of the outer quadrilateral?" Yes.

Page 82 follows this approach in pattern. The circle we study has a diameter of 6" in length. Ten points (labeled *a, b, c, d, e, f, g, h, i,* and *j*) are located —not evenly spaced—around the circle. By connecting these points, we construct an "inner polygon." Through each point we draw a line that, ideally, touches the circle only at that point. We label the points of intersections *A, B, C, D, E, F, G, H, I* and *J*, and connect them, thus constructing an "outer polygon."

The circumference is thus trapped between an "outer perimeter" and an "inner perimeter." We find the measure of each side of the outer perimeter, and of each side of the inner perimeter. Unless our measurements are grossly inaccurate, we find two measures—and feel confident that the measure of the circumference must lie somewhere between them.

Consider a small section:



The length of the line connection *x* and *z* is the shortest distance between two points, so is certainly less than the arc of the circle between *x* and *z*.

If a rubber band is stretched between *x* and *z*, we would have to stretch it further to lie along the arc of the circle from *x* to *z*, so we conclude that the arc is longer than the line from *x* to *z*. But we would have to stretch it even further if it is to reach from *x* to *y* to *z*. So we conclude that the sum of the distances *x* to *y* and *y* to *z* is greater than the distance along the arc between *x* and *z*. (Do you not use the word "proof" in this argument. We are appealing to intuition, to reasonableness—which though often a dangerous approach, is not at this point.)

Each student makes his own measurements to find the perimeter of the outer polygon and the inner polygon; and he records them in the tables, finding totals. The circumference must lie somewhere between. An average of the two is an approximation of the circumference (but not necessarily closer to the circumference than one of the perimeters).

Then a number is computed to complete the statement at the bottom of page 82. Finally, all the results found by the class are recorded and an average is computed. It will be quite close to 3.1416,

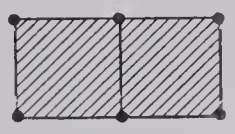
or 3 1/7, or 3 1/8. (Actually, 3 1/8 is a close approximation and convenient to use; 3 1/8 = 28/5 = 3.12 and its inverse, 8/25 = .32, is also more convenient than the usual 3 1/7. But tradition holds with the 3 1/7, even though it complicates computations.)

Page 83 recalls the number trapped in a narrow gap in the development of *Work Text* pages 74-77. The gaps are very nearly the same, and we suspect the actual number is the same. Mathematicians have found this to be true, and use π as its symbol.

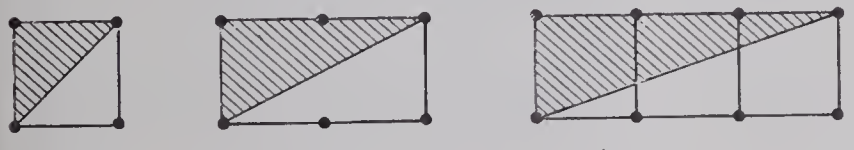
Notes to accompany *Work Text* pages 84-85

The next series of pages, beginning with 84 and concluding with 95, constitute an activity through which many students can discover the Pythogorean Theorem: The square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides. Students should *not* be told the outcome in advance. Keep the secret . . . all the way.

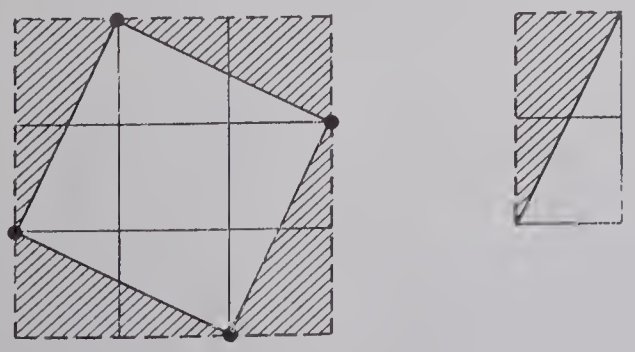
We start with a tile of a specified size. Here is a sketch of two such tiles:



How many tiles, cut at will, would be required to cover the shaded areas below:



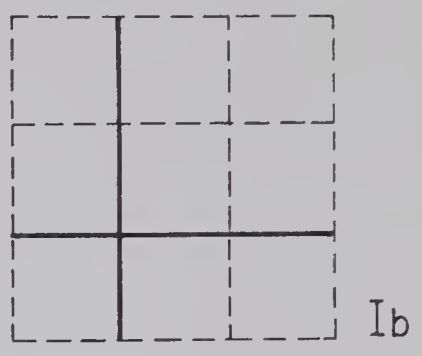
In *Ia*, students are asked to find several results. First, how many tiles would be required to cover the part of the sketch outside the square?



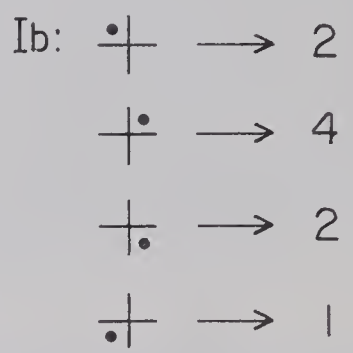
Each shaded corner would require half of two tiles, or one tile to cover it. The four corners together would require four tiles.

Later, the question is asked: "How many tiles are needed to cover the area inside the black square which is the unshaded area of the sketch above?" Since it would require nine tiles to cover the whole sketch and four of these would be needed for the shaded corners, that leaves five to cover the area inside the square.

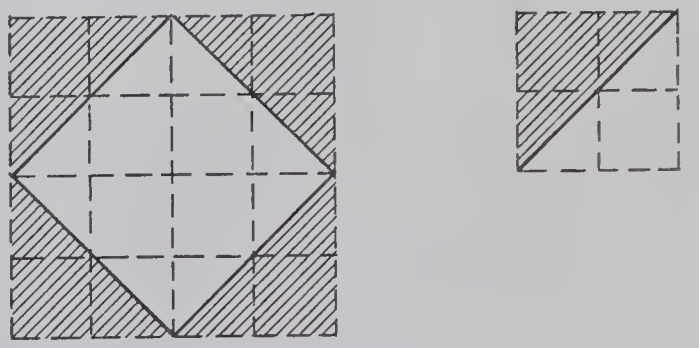
How many tiles in *Ia* are above the black line *and* to the left of the black line? Answer: two.



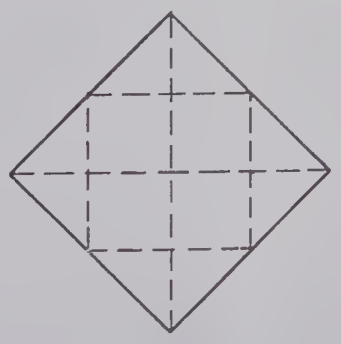
In this same symbolism, we would record:

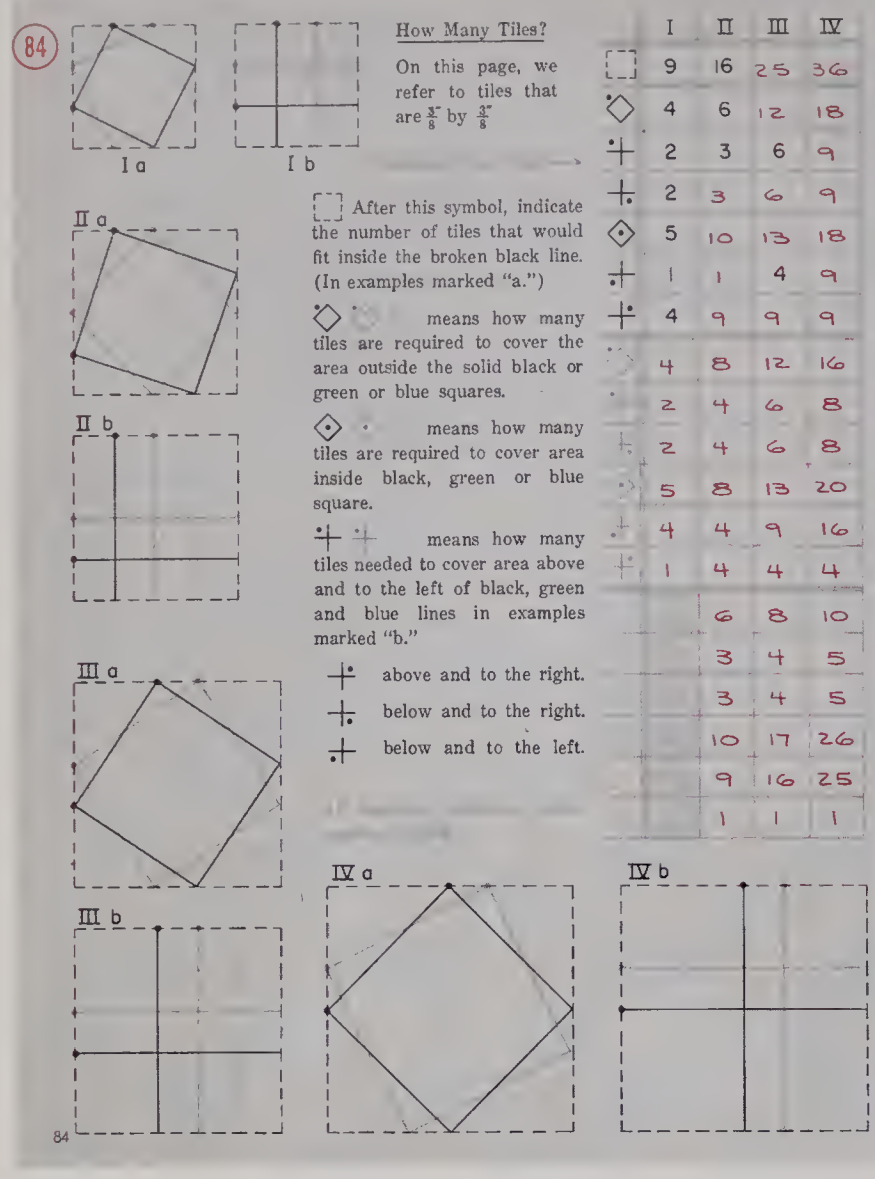


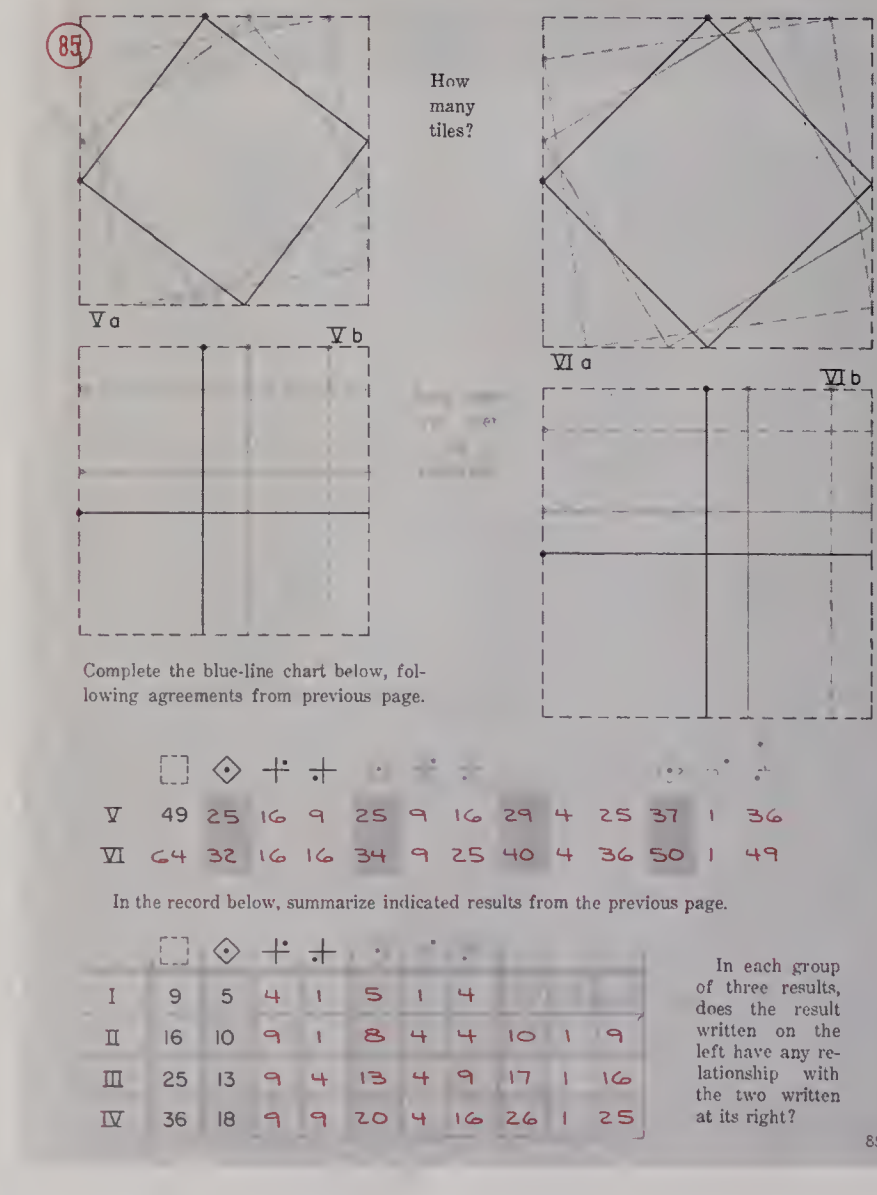
In sketch *IIa*, the green inscribed square presents this problem:



Each corner area shaded here is actually half of a four-tile area that is cut by a diagonal. So, it would require two tiles to cover each shaded corner —or eight tiles for the four corners. Since the whole sketch is composed of 16 tiles, there must be eight inside the green square.




84
How Many Tiles?
On this page, we refer to tiles that are 3/8" by 3/8"
I a
I b
II a
II b
III a
III b
IV a
IV b
After this symbol, indicate the number of tiles that would fit inside the broken black line. (In examples marked "a.")
means how many tiles are required to cover the area outside the solid black or green or blue squares.
means how many tiles are required to cover area inside black, green or blue square.
means how many tiles needed to cover area above and to the left of black, green and blue lines in examples marked "b."
above and to the right.
below and to the right.
below and to the left.
I II III IV
9 16 25 36
4 6 12 18
2 3 6 9
2 3 6 9
5 10 13 18
1 1 4 9
4 9 9 9
4 8 12 16
2 4 6 8
2 4 6 8
5 8 13 20
4 4 9 16
1 4 4 4
6 8 10
3 4 5
3 4 5
10 17 26
9 16 25
1 1 1
84


85
How many tiles?
V a
V b
VI a
VI b
Complete the blue-line chart below, following agreements from previous page.
V 49 25 16 9 25 9 16 29 4 25 37 1 36
VI 64 32 16 16 34 9 25 40 4 36 50 1 49
In the record below, summarize indicated results from the previous page.
I 9 5 4 1 5 1 4
II 16 10 9 1 8 4 4 10 1 9
III 25 13 9 4 13 4 9 17 1 16
IV 36 18 9 9 20 4 16 26 1 25
In each group of three results, does the result written on the left have any relationship with the two written at its right?
85

This seems reasonable because, if each of the four corners were folded into the center as in the sketch above, they would cover the area inside the square. There must be as much area outside the green square as inside. Of 16 tiles, it would require eight outside and eight inside.

Draw a sketch of each problem or each part of each example so that the reasoning is at all times completely convincing. Understanding of this idea is absolutely essential to understanding any of the development in the next 10 pages.

(Note: Since there is no blue used in *Ia* and *Ib,* that part of the record on the right is tinted blue and must, of course, be left blank.)

"Are there patterns in the charts on *Work Text* page 84?" Consider the charts, three lines at a time:

◇	4	6	12	18
·+	2	3	6	9
+·	2	3	6	9
	↑	↑	↑	↑

⊙	5	10	13	18
·+	1	1	4	9
+·	4	9	9	9
	↑	↑	↑	↑

Each number noted on the top line is the sum of the two numbers listed below it. "Does this pattern continue through the chart on page 84?" Yes. "On page 85, the chart is constructed differently. Can you find an equivalent pattern on page 85?" Yes.

25	16	9
32	16	16

25	9	16
34	9	25

29	4	25
40	4	36

37	1	36
50	1	49

The summary at the bottom of page 85 contains the same pattern.

Notes to accompany *Work Text* page 86

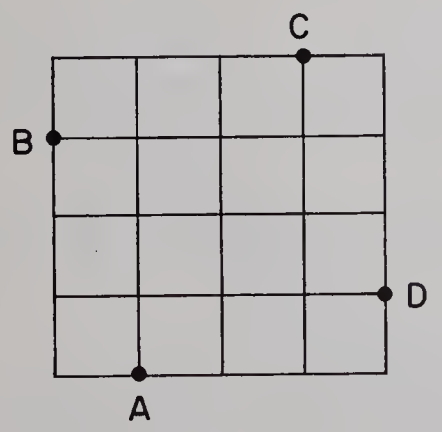
As the development proceeds through page 86, the outcome becomes more apparent to more children. The relationship between the top two rows in the chart and the bottom row is evident if the measurements are close. If the measurements are not done carefully, however, the idea behind the page will not be clear.

When everyone has results in the chart that reveal the relationship use a pattern indication. Suppose we agree that t suggests the number of inches in the side of the green square (top row in chart), m suggests the number of inches in the blue square (middle row in chart), and b suggests the number of inches in the side of the black square, then we note that:

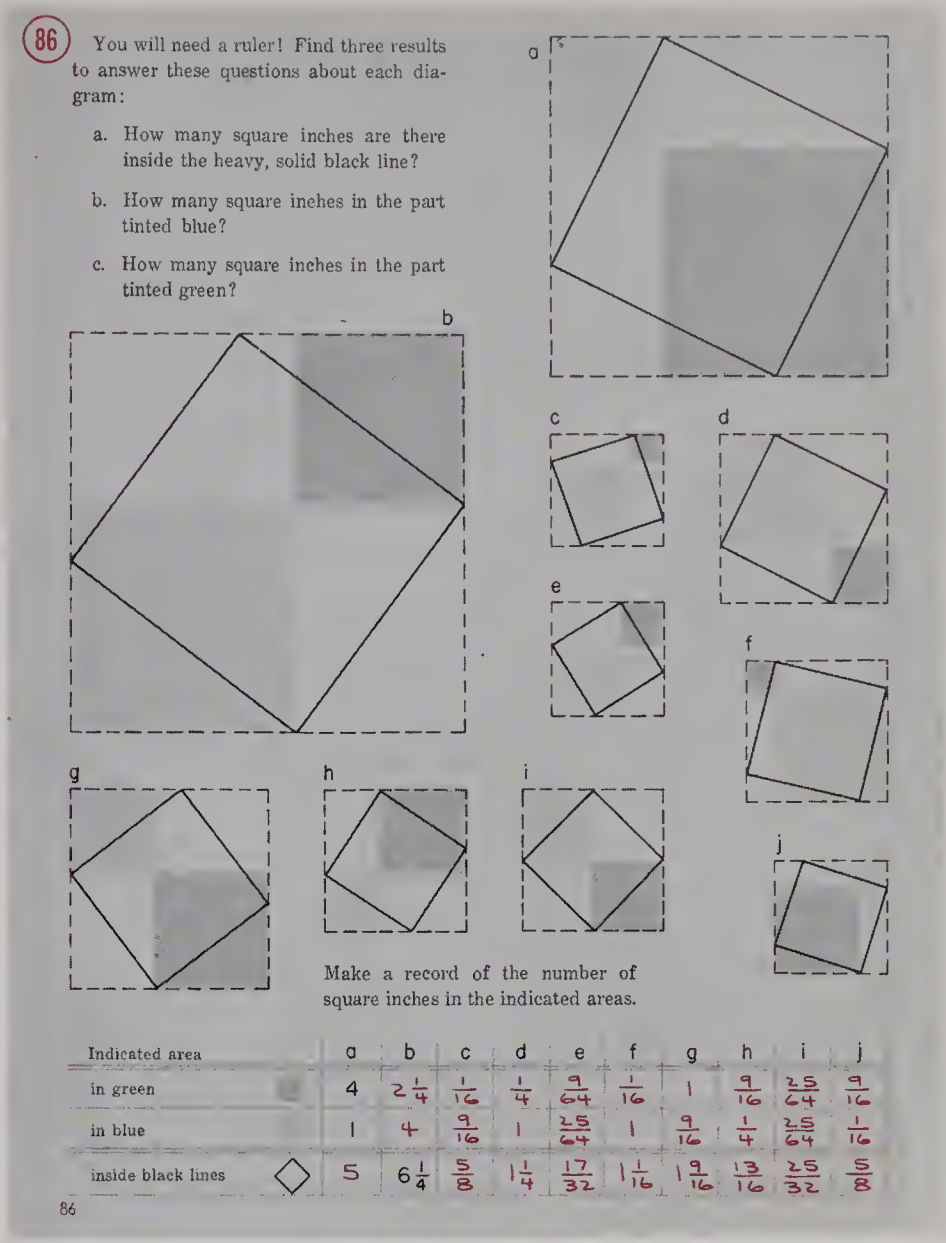
$$t \cdot t + m \cdot m = b \cdot b, \text{ or } t^2 + m^2 = b^2$$

Notes to accompany *Work Text* pages 87-89

Page 87 has a slight shift in emphasis to provide some rather intricate problems of measurements and to provide it in a context that will help reveal the firmness with which each student has grasped the idea of finding areas upon which this whole development turns. Pages 88 and 89 introduce a new way of describing each sketch. In every example, we inscribe one square inside another. In any square array of tiles, we completely determine the inscribed square by locating one corner of vertex.

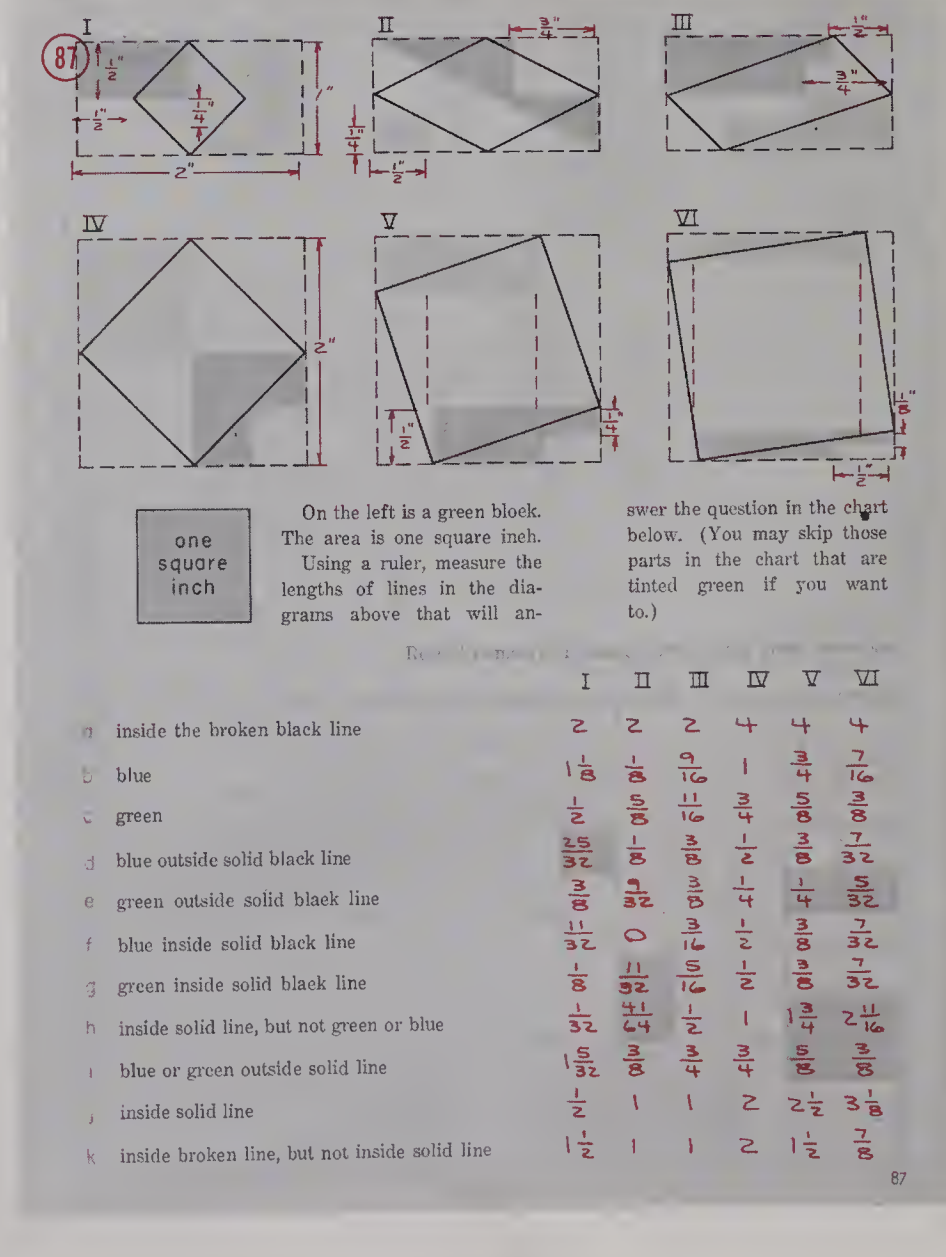


As soon as we locate point A, the points B, C and D are determined if A is one vertex of an inscribed square. In this case, A is the width of one square from the nearest vertex of the outside square, and the width of three squares from the next nearest vertex. So, there will be points the width of *one* square and the width of *three* squares from each vertex of the outer square.


86
You will need a ruler! Find three results to answer these questions about each diagram:
a. How many square inches are there inside the heavy, solid black line?
b. How many square inches in the part tinted blue?
c. How many square inches in the part tinted green?
a
b
c
d
e
f
g
h
i
j
Make a record of the number of square inches in the indicated areas.

Indicated area	a	b	c	d	e	f	g	h	i	j
in green	4	$2\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{9}{64}$	$\frac{1}{16}$	1	$\frac{9}{16}$	$\frac{25}{64}$	$\frac{9}{16}$
in blue	1	4	$\frac{9}{16}$	1	$\frac{25}{64}$	1	$\frac{9}{16}$	$\frac{1}{4}$	$\frac{25}{64}$	$\frac{1}{16}$
inside black lines ◇	5	$6\frac{1}{4}$	$\frac{5}{8}$	$1\frac{1}{4}$	$\frac{17}{32}$	$1\frac{1}{16}$	$1\frac{9}{16}$	$\frac{13}{16}$	$\frac{25}{32}$	$\frac{5}{8}$

86


87
I
II
III
IV
V
VI
2"
one square inch
On the left is a green block. The area is one square inch.
Using a ruler, measure the lengths of lines in the diagrams above that will answer the question in the chart below. (You may skip those parts in the chart that are tinted green if you want to.)

	I	II	III	IV	V	VI
a inside the broken black line	2	2	2	4	4	4
b blue	$1\frac{1}{8}$	$\frac{1}{8}$	$\frac{9}{16}$	1	$\frac{3}{4}$	$\frac{7}{16}$
c green	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{3}{8}$
d blue outside solid black line	$\frac{25}{32}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{7}{32}$
e green outside solid black line	$\frac{3}{8}$	$\frac{9}{32}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{32}$
f blue inside solid black line	$\frac{11}{32}$	0	$\frac{3}{16}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{7}{32}$
g green inside solid black line	$\frac{1}{8}$	$\frac{11}{32}$	$\frac{5}{16}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{7}{32}$
h inside solid line, but not green or blue	$\frac{1}{32}$	$\frac{41}{64}$	$\frac{1}{2}$	1	$1\frac{3}{4}$	$2\frac{11}{16}$
i blue or green outside solid line	$1\frac{5}{32}$	$\frac{3}{8}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{3}{8}$
j inside solid line	$\frac{1}{2}$	1	1	2	$2\frac{1}{2}$	$3\frac{1}{8}$
k inside broken line, but not inside solid line	$1\frac{1}{2}$	1	1	2	$1\frac{1}{2}$	$\frac{7}{8}$

87

88

On this and the following page we are going to experiment with square arrays of tile that are $\frac{1}{4}$" by $\frac{1}{4}$". Results will be recorded in the chart on the next page.

In a 2×2 array (I), a dot is marked in the lower left-hand corner. Mark two more dots — one, the width of a tile to the right and another, the width of a tile up. Connect these two dots.

Consider this line as the side of a square. How many tiles would be needed to cover the area inside? 2

Next, consider a 3×3 array (II). From a dot in the lower left-hand corner, locate two dots — two to the right and one up. Connect them, and consider the line as the side of a square. It will require 5 tiles to cover it.

In diagram III, follow the same procedure, except that the two dots are to be located the width of one tile to the right and two tiles up.

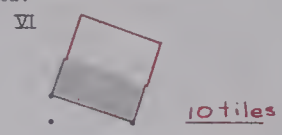
Finish drawing the square. How many tiles are required to cover it? 5

Did you expect the results in examples II and III to be the same?

Complete the following examples of 4×4 arrays:

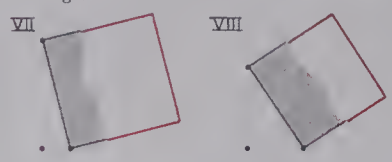
How many tiles are required to cover the squares drawn in the two examples of 4×4 arrays below left (IV and V)? 10 and 8.

Will the next example (VI) produce any new results, or will it repeat one already completed?



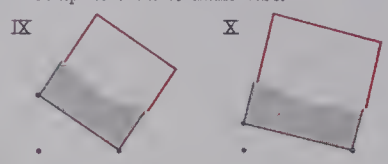
10 tiles

Next, consider two examples based on 5×5 arrays (VII and VIII). Complete the diagram.



How many tiles are required to cover each of these two squares? 17 and 13.

Do you think the next two examples (IX and X) will produce any new results? Complete them to make sure.



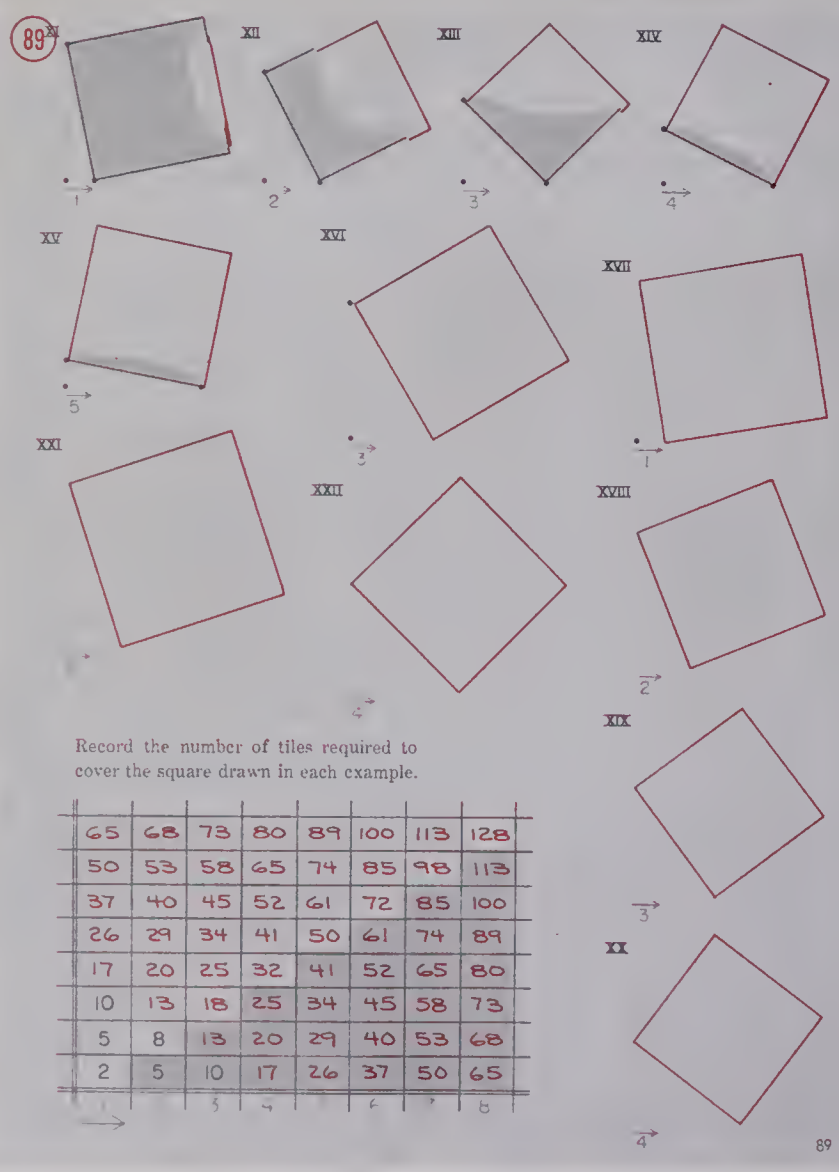
How many tiles are required to cover each of these two squares? 13 and 17.

On the chart on the next page, record all of your results.

Complete the diagrams on the next page in the pattern we have used on this page.

In the chart, what do you notice about the parts tinted blue and green?

Can you design your own experiment to complete the chart?



Record the number of tiles required to cover the square drawn in each example.

65	68	73	80	89	100	113	128
50	53	58	65	74	85	98	113
37	40	45	52	61	72	85	100
26	29	34	41	50	61	74	89
17	20	25	32	41	52	65	80
10	13	18	25	34	45	58	73
5	8	13	20	29	40	53	68
2	5	10	17	26	37	50	65

If the lower left-hand vertex of the outer square is taken as a reference point, we can refer to an inscribed square as "over 1 unit and up 3 units"—and each vertex of the outer square will follow that pattern.

The chart on *Work Text* page 89 (for pages 88 and 89) uses that idea: the record for each example is indicated by locating numbers in the green and blue sequences to indicate "over ____" and "up ____" respectively.

"If we made that into a 'What's My Rule?' game, and you called out a pair of numbers, I would find those numbers noted in the green and blue sequence of numbers and answer with the number noted in the cell above and to the left.

"For example: If you call out 1 and 1, I will answer 2; if you call out 3 and 1, I will answer 10; etc." Here is what a record would look like:

1, 1 → 2
3, 1 → 10
2, 2 → 8
2, 3 → 13
1, 2 → *5*
1, 3 → *7*
1, 4 → *9*
1, 5 → *11*

"Could anyone follow my rule if he lost the chart?"

If anyone can, let him. He may mentally, or on scratch paper, make a sketch similar to those on pages 88 and 89. If, however, he can anticipate the answer, he has probably seen the scheme: square each of the two numbers given and add their results. Give that sum as the response. If he uses this rule, swear him to secrecy. Let every child have the opportunity of discovering the pattern. There are more opportunities provided in the following pages.

Notes to accompany *Work Text* pages 90-93

We begin by recording results from previous investigations in a chart—and find one section filled. There are several rows and columns and diagonals that have at least three or four consecutive cells completed.

The main diagonal is the first considered. In past pattern hunts we have found important clues by noting the differences between successive terms of a sequence. The difference in the series noted in the main diagonal begins

6, 10, 14

and the differences between the terms of this beginning suggest that the bottom line will continue to repeat "4." If that is the case, we can keep a chain reaction going as far as we please.

But we can only hope that this pattern is a reliable way to predict results as the chart is extended. So, tests are called for. Example XXIII is such a test—and the results add to our confidence that we have uncovered a good predictor. On the board, extend this test to the next term—and, the prediction remains a good one.

At the bottom of *Work Text* page 90, three more entries for the record above are worked out.

Next, consider having small groups of students tackle page 91 with help from you when necessary. When most have completed page 91, have the groups compare results. When general agreement has been reached, discuss the final question: the pattern in the chart's diagonals, down-and-right. For example:

26, 20, 18, 20, 26

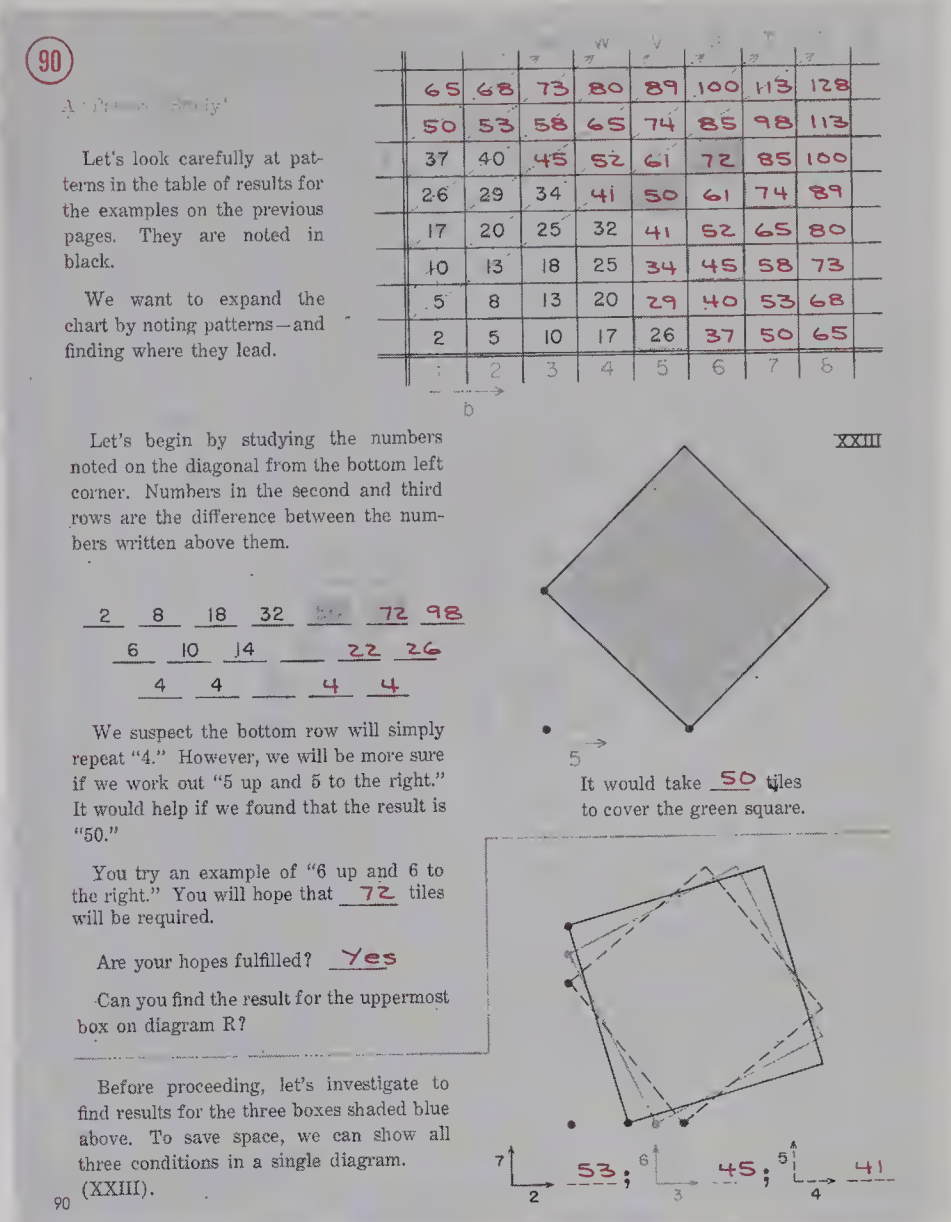
is obvious, but how can it apply in the next diagonal above?

37, 29, 25, 25, ___, ___

You might refer to this pattern as a mirror image in the sense that on each side of the main diagonal, up and right, these sequences repeat themselves in reverse order. The pattern is very much like one set of the diagonals of the "multiplying machine" that is an extension of:

4	0	4	8	12	16
3	0	3	6	9	12
2	0	2	4	6	8
1	0	1	2	3	4
0	0	0	0	0	0
X	0	1	2	3	4

Work Text page 92 works toward pattern indicators that seem to be repeating themselves often in this development. You might consider preceeding the work on page 92 with several "What's My Rule?" games that suggest the idea in those summarized in the lower section of page 92.



90

Let's look carefully at patterns in the table of results for the examples on the previous pages. They are noted in black.

We want to expand the chart by noting patterns—and finding where they lead.

	65	68	73	80	89	100	113	128
	50	53	58	65	74	85	98	113
	37	40	45	52	61	72	85	100
	26	29	34	41	50	61	74	89
	17	20	25	32	41	52	65	80
	10	13	18	25	34	45	58	73
	5	8	13	20	29	40	53	68
	2	5	10	17	26	37	50	65
	1	2	3	4	5	6	7	8

b →

Let's begin by studying the numbers noted on the diagonal from the bottom left corner. Numbers in the second and third rows are the difference between the numbers written above them.

2 8 18 32 ___ 72 98

6 10 14 ___ 22 26

4 4 ___ 4 4

We suspect the bottom row will simply repeat "4." However, we will be more sure if we work out "5 up and 5 to the right." It would help if we found that the result is "50."

You try an example of "6 up and 6 to the right." You will hope that 72 tiles will be required.

Are your hopes fulfilled? Yes

Can you find the result for the uppermost box on diagram R?

Before proceeding, let's investigate to find results for the three boxes shaded blue above. To save space, we can show all three conditions in a single diagram. (XXIII).

XXIII

It would take 50 tiles to cover the green square.

90

91

Now you can fill in all the blue-line blanks from the chart (top row) or by finding the difference of the two numbers written above it (second and third rows):

→ S

5 13 25 41 61 85 113

8 12 16 20 24 28

4 4 4 4 4

10 20 34 52 74 100 130

10 14 18 22 26 30

4 4 4 4 4

17 29 45 65 89 117 149

12 16 20 24 28 32

4 4 4 4 4

26 40 58 80 106 136 170

14 18 22 26 30 34

4 4 4 4 4

→ W

37 53 73 97 125 157 193

16 20 24 28 32 36

4 4 4 4 4

Diagonal "X" has no results given. The pattern you see will suggest results for the second and third rows.

? 50 68 90 116 146 180 218

18 22 26 30 34 38

4 4 4 4 4

If we know only the first number (on the left) in the top row, we could complete the whole example.

Perhaps a similar study of the left-hand column of results will help. This column is composed of the first number in each of the diagonals we have been studying.

2 5 10 17 26 37 50

3 5 7 9 11 13

2 2 2 2 2

Now you can complete → "X."

Did the results in the bottom row surprise you? Let's try the second column from the left in our record.

5 8 13 20 29 40 53 68 85 104

3 5 7 9 11 13 15 17 19

2 2 2 2 2 2 2 2

That is monotonously like the previous study. Try the third column from the left.

10 13 18 25 34 45 58 73 90 109

3 5 7 9 11 13 15 17 19

2 2 2 2 2 2 2 2

No longer interesting!

How can you quickly fill in all the spaces below and to the right of the diagonal?

Using the patterns you already know, you can quickly make a chart as large as you want.

Notice the "over one and up two" diagonal of green boxes. Find more in this series?

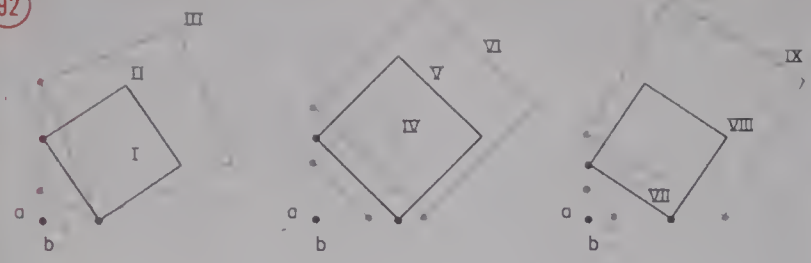
26 45 74 113 162 221 290

19 29 39 49 59 69

10 10 10 10 10

91

92



How many tiles would be required to cover each of the three squares in each of the examples above?

For the last four examples, you may need to draw your own sketches—unless you've discovered a short cut.

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII
a ↑ ... distance up	1	3	5	2	3	4	1	2	3	2	3	3	4
b → distance to right	2	2	2	2	3	4	1	3	5	5	2	3	3
tiles required to cover squares	5	13	29	8	18	32	2	13	34	29	13	18	25

"Call out any two numbers," Mrs. Boswell said. "I will apply each of my rules and write down my results. If you discover the rules, then you know in advance what I am going to write."

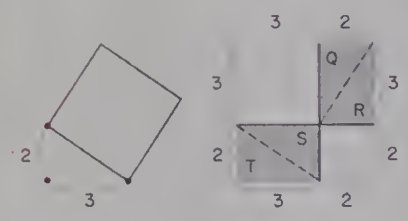
Below are some of the results:

a	b		a	b		a	b	
3	4	→ 9	3	4	→ 16	3	4	→ 25
2	7	→ 4	2	7	→ 49	2	7	→ 53
5	5	→ 25	5	5	→ 25	5	5	→ 50
8	6	→ 64	8	6	→ 36	8	6	→ 100
9	1	→ 81	9	1	→ 1	9	1	→ 82
10	5	→ 100	10	5	→ 25	10	5	→ 125
$a, b \rightarrow a \cdot a$ or a^2			$a, b \rightarrow b \cdot b$ or b^2			$a, b \rightarrow a^2 + b^2$		
7	4	→ 49	7	4	→ 16	7	4	→ 65
6	9	→ 36	6	9	→ 81	6	9	→ 117
7	8	→ 49	7	8	→ 64	7	8	→ 113
5	12	→ 25	5	12	→ 144	5	12	→ 169

92

93

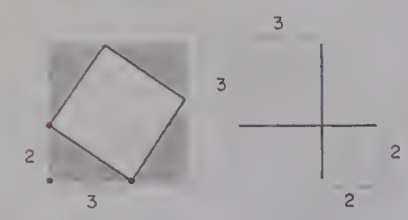
Here is a familiar example: "up 2 and over 3" . . . and a companion grid the same over-all size.



There are 25 tiles in each 5 × 5 array. On the right above, the illustration has certain spaces lettered so we can keep track of them.

Suppose we cut out the pieces tinted green and cut each along the broken line.

We can paste these pieces on top of the diagram on the left. The finished diagram would then appear as below?



From the 5 × 5 grid on the right, we removed 2 · 3 tiles and then 2 · 3 more. Or we removed 12 of the 25 tiles.

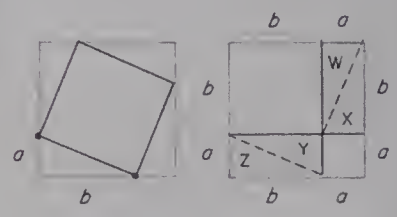
These 12, when cut to fit, will fill the corners outside the tilted square. They cover 12 of the original 25 tiles.

We have left 2 · 2 and 3 · 3 tiles, 4 + 9 = 13. And, if cut to fit, these 13 tiles certainly ought to cover the tilted square itself.

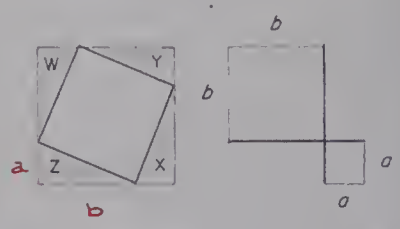
Do you agree?

But there is no longer any charm about that "2 up and 3 over." Think of any two numbers: we will call them "a" and "b" so we can keep track of them—even though we don't know what they are.

Now we draw "a tiles up and b over."



Next we draw in the broken lines and cut out the pieces tinted blue—and cut them on that broken line. After we paste them on the sketch at the left, our results are:



We conclude that $b \cdot b$ tiles and $a \cdot a$ tiles together will be needed to cover the inside of the tilted square in the sketch on the left above. That is $a^2 - b^2$ tiles.

If the two numbers (a and b) that someone thought of are indicated below, you can find the number of tiles needed to cover the square.

	I	II	III	IV	V
a	2	5	7	10	3
b	3	5	3	8	4
$a^2 + b^2$	13	50	58	164	25

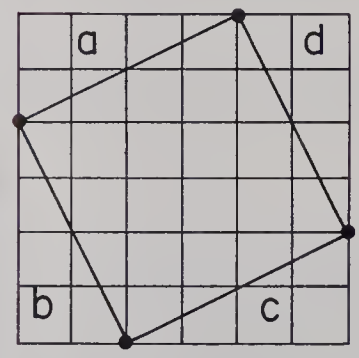
93

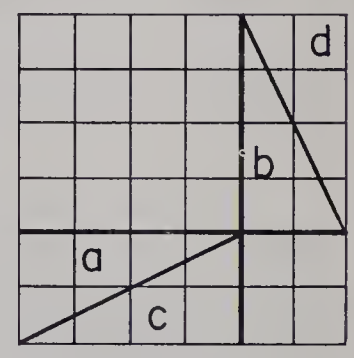
2, 8	→ 16		9, 3	→ 3	
19, 5	→ 10		12, 17	→ 4	
0, 7	→ 14		6, 13	→ 5	
11, 25	→ 50		____, ____	→ ____	
____, ____	→ ____		____, ____	→ ____	
____, ____	→ ____				
a, b	→ $2b$		a, b	→ $a/3$	
____, ____	→ ____		____, ____	→ ____	
____, ____	→ ____		____, ____	→ ____	

If someone asks, "How can you drop out part?" we could agree to write a longer pattern indicator if that would help:

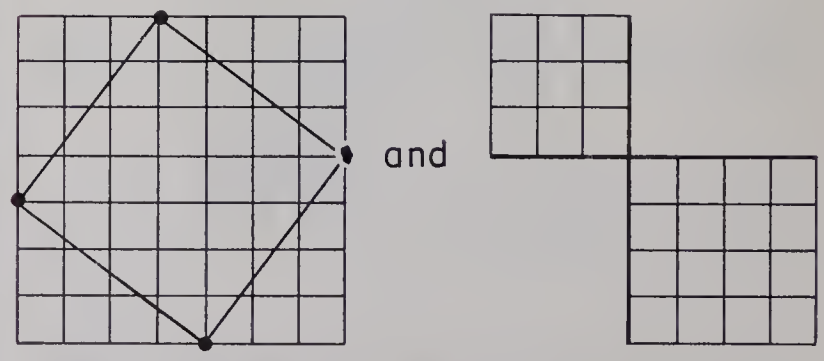
$a,b \rightarrow 0 \cdot a + 2b$ and
$a,b \rightarrow a/3 + 0 \cdot b$

The example worked out in detail on page 93 ought to be repeated several times with enlarged diagrams made out of construction paper:

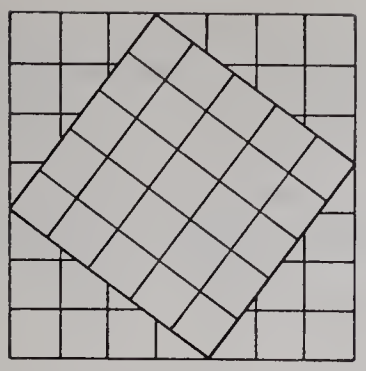




Actually cut out the pieces as shown on the right and paste the triangle on top of the piece on the left, filling the corners—and leaving the two squares on the right. Among the demonstrations, include a 7 by 7 grid, illustrating "3 up and 4 across:"



When this point has been reached, suggest that someone cut up the two squares on the right and paste them on the large square on the left. Because $3^2 + 4^2 = 5^2$, the results are:

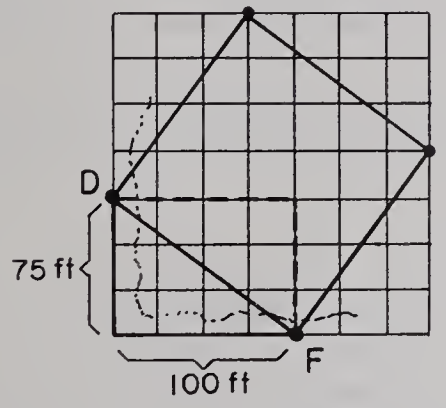


When growing knowledge demonstrates its usefulness by helping solve a problem that looks impossible at first glance, it adds to the conviction that it is a worthwhile body of knowledge.

"How can Fred and Dan find the distance between two points with a corner of a lake in between—without getting wet and without a boat available?"

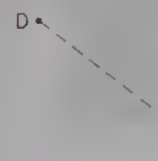
Quite a feat! But their procedures are outlined on page 94.

When the sketch at the top of the right-hand column is studied, consider adding lines to a similar sketch on the board so that it becomes a 7 unit by 7 unit grid:



It is evident that each unit is a square area, 25 feet by 25 feet, containing 625 square feet. This may help students see that each corner of the sketch has an area of 1/2 of a rectangle 75 feet by 100 feet—3,750 square feet. This result can be checked by noting that the rectangle is 3 units × 4 units, or 12 square units, with 6 on each side of the diagonal. Since each square unit represents 625 feet and 6 × 625 = 3,750, there must be 3,750 square feet in each corner—which checks with the other result.

The 7 by 7 grid can be recalled when reference is made to the early Egyptian's rope with evenly spaced knots. The right triangle they can form will be 3 units by 4 units by 5 units. And, since we find that the distance between D and F (in the sketch on page 94) is 125 feet, we recognize this as the same as 5 units.



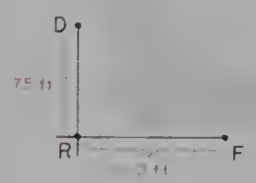
Dan and Fred want to stretch a wire between their homes so they can operate telephones they have made themselves.

How much wire will they need? They can't measure the distance directly, because the wire will cross a lake (as shown in the diagram above) from Dan's house, "D," to Fred's house, "F".

They went about measuring the distance in this way.

They tied a string to the post in front of Dan's house, and another to a post in front of Fred's house.

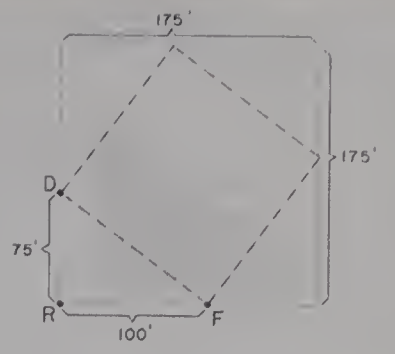
It required considerable experiment but they finally got them to cross at point "R" (below) so that the two strings made a square corner. Like this:



The length of the string from F to R was measured—on dry land. It was 100 ft. long. It took 75 ft. of string to reach from D to R—on dry land.

But they want to know how far it is from D to F—over the water.

So, they made a sketch very much like the sketches we have been studying.



Inside the green square, how many square feet would there be?

175 x 175 = 30,625

Inside each corner, how many square feet would there be?

$\frac{1}{2}$ x 75 x 100 = 3,750

In four such corners, how many square feet would there he?

4 x 3,750 = 15,000

Then, inside the tilted square indicated by the broken line, how many square feet would there be?

30,625 - 15,000 = 15,625

What is the length of each side of a square with an area of that many square feet?

15,625 = 125 x 125

Hint

100 x 100 = 10,000 120 x 120 = 14,400
110 x 110 = 12,100 130 x 130 = 16,900

So the numbers we are looking for must be more than 120 and less than 130.

94

Notes to accompany *Work Text* pages 94-95

On these two pages, we consider several instances in which the three sides of the right triangle are all whole numbers of units. Perhaps some children would like to find others:

Side a		Side b		Side c	
Units	Square of units	Units	Square of units	Units	Square of units
3	9	4	16	5	25
20	400	21	441	29	841
75	5625	100	10000	125	15625
8	64	15	225	17	289

There are several ways to find more examples.

A) Any group, such as 3, 4 and 5 in which

$3^2 + 4^2 = 5^2$ so will 2(3), 2(4) and 2(5)
$6^2 + 8^2 = 10^2$ (36+64 = 100)
$9^2 + 12^2 = 15^2$ (81+144 = 225)
or
$8^2 + 15^2 = 17^2$ (64+225 = 289)
$16^2 + 30^2 = 34^2$
or
$20^2 + 21^2 = 29^2$ (400+441 = 841)
$10^2 + 10\ 1/2^2 = 14\ 1/2^2$
$5^2 + 4\ 1/4^2 = 7\ 1/4^2$

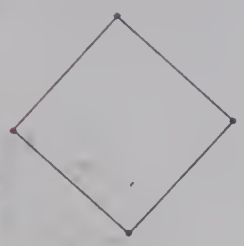
95 Marianne made an interesting device using three stakes and three lengths of string. (Stakes are labeled X, Y and Z in following sketch:)

The string from X to Y measured 20 ft. when stretched out straight. The piece between X and Z measured 21 ft., and the piece from Y to Z measured 29 ft.

Marianne made the following claim:

Do you believe she is right?

She explained her device this way:

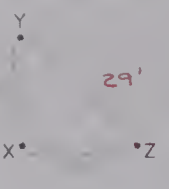


She put some arithmetic on the board and let the class complete it.

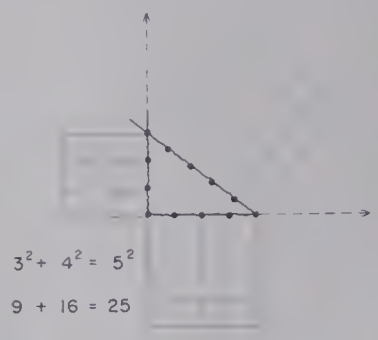
20 x 20 = 400 and 21 x 21 = 441

400 + 441 = 841

841 = 29 x 29



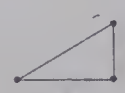
Early Egyptians used a similar device for working out square corners for their fields. They put knots in a rope the same distance apart. One side was 3 such distances, another 4, and the other was 5. They thought about it this way:



$3^2 + 4^2 = 5^2$

9 + 16 = 25

They argued this way: We think of a triangle with sides that are "a," "b" and "c"; and, if $a^2 + b^2 = c^2$, then the triangle has a square corner and is called a "right triangle."



If this is true, is a triangle whose sides are 15", 8" and 17" a "right triangle"? . . . or does $15^2 + 8^2 = 17^2$? Yes

225 + 64 = 289 = 17·17

95

B) Take any odd number, say 9. Square it (81). Split the square into two whole numbers as nearly equal as you can (40, 41). Then 9, 40 and 41 have the relationship we are concerned with:

$9^2 + 40^2 = 41^2$ (81+1600 = 1681)

Begin with 5. Square it: 25. Split into two whole numbers as nearly equal as possible: 12 and 13.

$5^2 + 12^2 = 13^2$ (25+144 = 169)

C) Some pupils may want to try this:

(x, y)	2xy	$x^2 - y^2$	x + y
(4, 3)	$2 \cdot 4 \cdot 3 = 24$	$4^2 - 3^2 = 7$	16 + 9 = 25
(5, 2)	20	21	29
(9, 3)	54	72	90

And some of the students might like to experiment with these rules. For example, do they always supply examples in the relationship $a^2 + b^2 = c^2$?

Notes to accompany *Work Text* pages 96-97

A game of "What's My Number?" might well introduce *Work Text* pages 96 and 97.

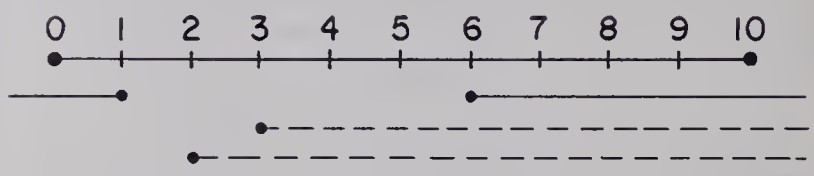
"I'm thinking of a number. It is not less than 0 nor more than 10. You call out a number somewhere in between. I will tell you if my number is greater than or less than the one you call out. We will keep a record on the number line." (Why doesn't the line extend to the left of 0 and to the right of 10 in the sketch below?)

Bill calls, "Six."

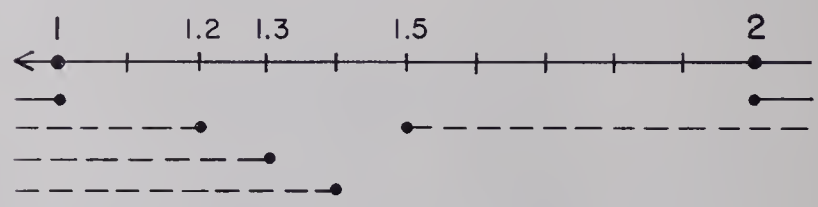
"My number is less than 6."

Alice calls, "One."

"My number is more than 1."



It must be between 1 and 6. Suppose your next two answers are, "My number is less than 3," and "My number is less than 2." Your number must be between 1 and 2. We will magnify that section of the record:



"One and two-tenths!"

"My number is more than 1.2."

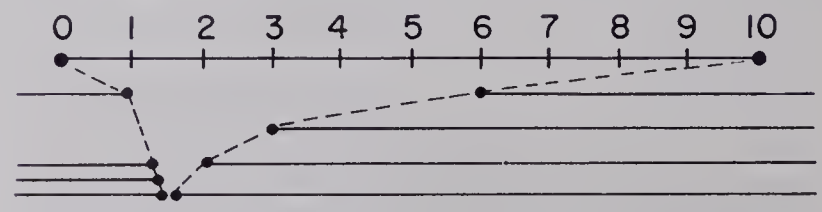
"One and five-tenths!"

"My number is less than 1.5."

"One and four-tenths!"

"My number is more than 1.4."

So, the squeeze is on. It has been discovered that your number is more than 1.4, but less than 1.5. Here is a summary:



What is your number? Each time a number is called out, you find a product with that number taken as a factor twice. If the product is more than 2, you

say: "My number is less than the one you called out." If it is less than 2, you say: "My number is more than the one you called out." Reveal your rule. Test the last plays:

$1.4 \times 1.4 = 1.96$ . . . "My number is more than 1.4."

$1.5 \times 1.5 = 2.25$ . . . "My number is less than 1.5."

Let someone else take over your part in the game. Keep the search going at least until your number is trapped between 1.41 and 1.42.

Change the game. This time the rule is: when your number is taken twice as a factor, the product is 5. It is more than 2 and less than 6, because:

$2 \times 2 = 4$ and $3 \times 3 = 9$

It must be between 2.2 and 2.3, because:

$2.2 \times 2.2 = 4.84$ and $2.3 \times 2.3 = 5.29$

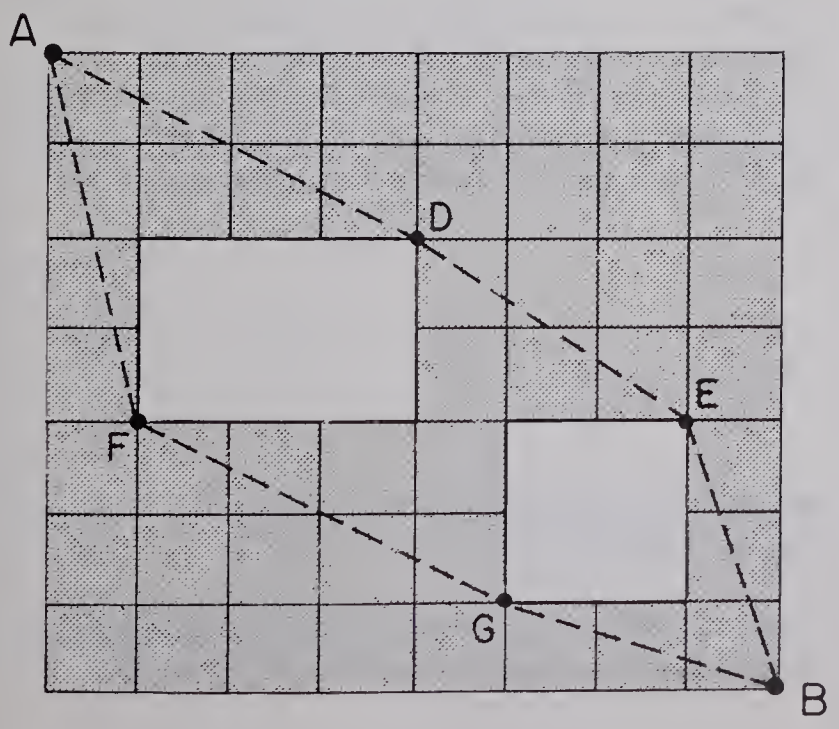
It can be found that your number must be between 2.23 and 2.24 because:

$2.23 \times 2.23 = 4.9729$ and $2.24 \times 2.24 = 5{,}0176$

With these introductory games, pages 96 and 97 can be completed by most students independently.

A puzzle

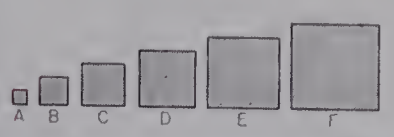
You might like to challenge the class with this puzzle: A piece of graph paper has a square and a rectangle cut out of it as indicated in the sketch below. An ant wants to take the shortest route from the corner labeled *A* to the corner labeled *B*. What route should he choose? (We will refer to the length of a side of each small square as a unit.)



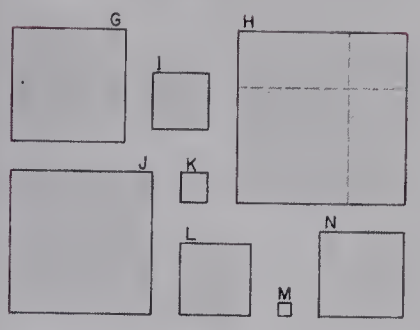
96

If you know the area of a square, can you find the length of its sides?

In the diagram below, 1" represents 1 foot.



	A	B	C	D	E	F
Area in sq. ft.	1	4	9	16	25	36
Sides in feet	1	2	3	4	5	6



Record the length of the sides in numbers of inches and the areas in number of square inches in the above examples.

	G	H	I	J	K	L	M	N
Length of sides	1	$1\frac{1}{2}$	$\frac{1}{2}$	$1\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{8}$	$\frac{1}{8}$	$\frac{3}{4}$
Area	1	$2\frac{1}{4}$	$\frac{1}{4}$	$1\frac{9}{16}$	$\frac{1}{16}$	$\frac{25}{64}$	$\frac{1}{64}$	$\frac{9}{16}$

In these last examples, you measured the length of the sides and then computed the area.

Suppose that you are told that the area of a square is 2 square inches. What is the length of the sides?

This is the same as asking for a number which, when multiplied by itself, is equal to 2; or:

$S \cdot S = S^2 = 2$

Is the number we are looking for more or less than 1? more

Is the number we are looking for more or less than 2? less

So, we know that it is more than one and less than 2 because:

$1 \times 1 = 1$ and $2 \times 2 = 4$

Complete the following:

$1.1 \times 1.1 =$ 1.21	$1.5 \times 1.5 =$ 2.25
$1.2 \times 1.2 =$ 1.44	$1.6 \times 1.6 =$ 2.56
$1.3 \times 1.3 =$ 1.69	$1.7 \times 1.7 =$ 2.89
$1.4 \times 1.4 =$ 1.96	$1.8 \times 1.8 =$ 3.24

Now we know that the number we are looking for is more than 1.4 and less than 1.5, and you might guess that is closer to the smaller number.

Let's find out more by completing the following computations.

$1.41 \times 1.41 =$ 1.9881 $1.43 \times 1.43 =$ 2.0449

$1.42 \times 1.42 =$ 2.0164 $1.44 \times 1.44 =$ 2.0736

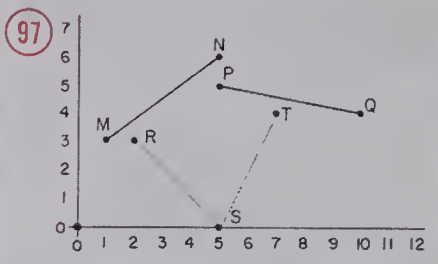
Now, we know that the number we are looking for is more than 1.41 and less than 1.42.

One number is .0119 too small and the other is .0164 too large. What conclusions can you draw from the following?

$1.414 \times 1.414 =$ 1.999396

$1.415 \times 1.415 =$ 2.002225

97



One approach might be to think of the lines in this way.

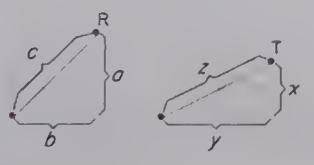
$\overline{MN}$ (read "line segment Em En") reaches between opposite corners of a 3×4 array of square units.

$\overline{PQ}$ reaches between opposite corners of a 1×5 array; $\overline{RS}$ between opposite corners of a 3×3 array; and $\overline{ST}$ between opposite corners of a 2×4 array.

All arrays the same shape and size would be as good as any other, so we can redraw the lines all starting from a single point — labeling only one end to identify the line.

Both black lines seem to be longer than the green lines. Which black line is longer? Which green line is shorter?

Consider the green lines:



Our earlier experiments and study led us to a conclusion we can draw about the situation in these sketches:

$a^2 + b^2 = c^2$ or $9 + 9 = 18 = c^2$

$x^2 + y^2 = z^2$ or $4 + 16 = 20 = z^2$

$4^2 =$ 16 and $5^2 =$ 25

So "c" and "z" are each more than 4 but less than 5. So, follow the procedure used on the previous page.

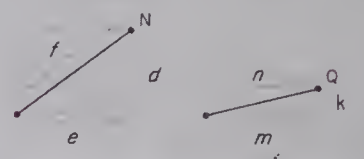
$4.1 \times 4.1 =$ 16.81	$4.4 \times 4.4 =$ 19.36
$4.2 \times 4.2 =$ 17.64	$4.5 \times 4.5 =$ 20.25
$4.3 \times 4.3 =$ 18.49	$4.6 \times 4.6 =$ 21.16

We can see that "c" must be more than 4.2 but less than 4.3.

And, we see that "z" must be more than 4.4 but less than 4.5.

Therefore, line segment $\overline{RS}$ is less (more or less) than line segment $\overline{ST}$.

Next, consider the black lines.



We can conclude that:

$d^2 + e^2 = f^2$ or $9 + 16 = 25$

$k^2 + m^2 = n^2$ or $1 + 25 = 26$

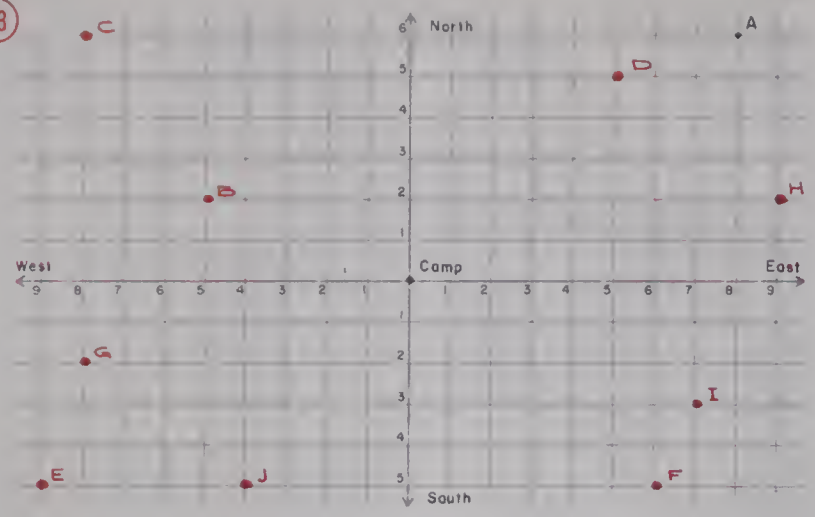
$f^2 = 25$ so $f = 5$.

$n^2 = 26$ so n is more (more or less) than 5.

$5 \times 5 =$ 25 and $5.1 \times 5.1 =$ 26.01

We know that "n" is more than 5 and less than 5.1.

Conclusion: In our original sketch, line segment $\overline{RS}$ is the smallest and line segment $\overline{RQ}$ is the largest.



A group of 10 prospectors scattered from their camp. After a week's work, each reported his location with respect to the camp.

The radio operator at the camp made a map and marked the position of each man by placing a black dot and labeling it with the first letter of the prospector's name. The operator's notes indicated these positions:

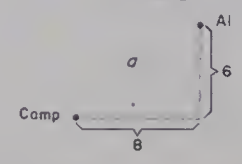
Al: 8 miles east and 6 miles north
Ben: 5 miles west and 2 miles north
Carl: 8 miles west and 6 miles north
Dan: 5 miles east and 5 miles north
Ed: 9 miles west and 5 miles south
Frank: 6 miles east and 5 miles south
Gil: 8 miles west and 2 miles south
Harry: 9 miles east and 2 miles north
Ira: 7 miles east and 3 miles south
Jack: 4 miles west and 5 miles south

Locate each prospector on the map and label the point as the operator did.

The director of the prospecting expedition asked the operator, "How far is Al from camp?"

"Al is 8 miles east and 6 miles north. I'll tell you in a minute the shortest distance by plane."

He drew the following sketch:



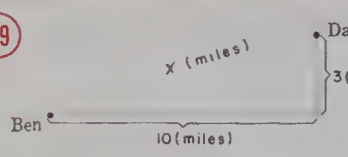
$a^2 = 8^2 + 6^2 = 64 + 36 = 100$

$a = 10$ (because $10 \times 10 = 100$)

"Al is 10 miles from camp," the radio operator advised.

"How far is Ben from Dan?"

"Well, Ben is 10 miles west of Dan and 3 miles south of him. I can tell you in a minute the distance between them — as a plane flies."



$x^2 = 3^2 + 10^2 = 9 + 100 = 109$

Now the operator has a problem. He must look for a number which, when multiplied by itself, or taken as a factor twice, is equal to 109.

$10 \times 10 = 100$ and $11 \times 11 = 121$

10 11

$10.5 \times 10.5 = 110.25$

less

$10.3 \times 10.3 = 106.09$ $10.4 \times 10.4 = 108.16$

10.4 10.5

$10.43 \times 10.43 = 108.7849$
$10.44 \times 10.44 = 108.9936$
$10.45 \times 10.45 = 109.2025$
$10.46 \times 10.46 = 109.4116$

10.44 10.45

All distances are recorded as numbers of miles. 10.44 + means more than 10.44 miles, but less than 10.45 miles apart.

	Al	Ben	Carl	Dan	Ed	Frank	Gil	Harry	Ira	Jack
Al	0	13.60+	16	3.16+	20.24+	11.18+	17.88+	4.12+	9.05+	16.27+
Ben	13.60+	0	5	10.44+	8.06+	13.03+	5	14	13	7.07+
Carl	16	5	0	13.03+	11.04+	17.80+	8	17.46+	17.49+	11.70+
Dan	3.16+	10.44+	13.03+	0	17.20+	10.04+	14.76+	5	8.24+	13.45+
Ed	20.24+	8.06+	11.04+	17.20+	0	15	3.16+	19.31+	16.12+	5
Frank	11.18+	13.03+	17.80+	10.04+	15	0	14.31+	7.61+	2.23+	10
Gil	17.88+	5	8	14.76+	3.16+	14.31+	0	17.46+	15.03+	5
Harry	4.12+	14	17.46+	5	19.31+	7.61+	17.46+	0	5.38+	14.76+
Ira	9.05+	13	17.49+	8.24+	16.12+	2.23+	15.03+	5.38+	0	11.18+
Jack	16.27+	7.07+	11.70+	13.45+	5	10	5	14.76+	11.18+	0
to →	Al	Ben	Carl	Dan	Ed	Frank	Gil	Harry	Ira	Jack

The broken lines indicate the two most likely alternatives. Label certain points so we can speak of these two routes: A D E B and A F G B.

Notes to accompany *Work Text* pages 98-99

After the first examples on *Work Text* page 98, in which it turns out that Al is 10 miles from camp, we are confronted with a problem in the pattern:

□ × □ = 109

and our rule is that the number written in one box must be written in both.

We know that the number is more than 10 (because $10 \times 10 = 100$, and 100 is less than 109) and less than 11 (because $11 \times 11 = 121$, and 1212 is more than 109). Thus, we are launched on another game of trapping a number into a narrowing gap.

As we approach most of the questions asked by the chart on page 99—how far is Jack from Al, Carl from Ed, etc.?—there is no whole number that will express the number of miles the prospectors are from each other.

Some students may want to tackle the whole task independently. However, there is a very considerable amount of computations involved—too much for most to handle. You might consider separating the class into five groups and let each tackle two rows of work. (Because the upper right and lower left sections are repetitions of each other, this means that each computation would be assigned to two groups and, at the conclusion, these would provide a check, one against the other. The chart repeats itself on each side of the diagonals of zeros.)

Notes to accompany *Work Text* pages 100-101

A pair of "What's My Rule?" games will help introduce the idea behind another bit of shorthand:

25 → 5	5 → 25
100 → 10	7 → 49
49 → 7	9 → 81
9 → 3	10 → 10
16 → ___	4 → ___
36 → ___	6 → ___
400 → ___	15 → ___
$x \rightarrow \sqrt{x}$	$x \rightarrow x^2$
___ → ___	___ → ___
___ → ___	___ → ___

A series of statements will add a bit of understanding for some:

$\sqrt{49} = 7$ $\qquad$ $2 \times \sqrt{9} = 6$

$8 = \sqrt{64}$ $\qquad$ $\frac{\sqrt{25}}{5} = 1$

$8^2 = 64$ $\qquad$ $\sqrt{4} + \sqrt{9} = \sqrt{25}$

$36 = 6^2$ $\qquad$ $\sqrt{16} \times \sqrt{16} = 16$

$\sqrt{36} = 6$ $\qquad$ $\frac{\sqrt{64}}{\sqrt{16}} = \frac{8}{4} = 2$

$\sqrt{6^2} = 6$ $\qquad$ $6 = \sqrt{100} - \sqrt{64}$

$\sqrt{36} = \sqrt{100} - \sqrt{64}$

"Are these *all* true statements?" Certain patterns are suggested by some of these examples. They ought to be tested in several similar expressions:

$\sqrt{3} \times \sqrt{2} = \sqrt{6}$

$\sqrt{9} \times \sqrt{25} = \sqrt{9 \times 25} = \sqrt{225}$

$3 \times 5 = 15$

$\sqrt{36} \times \sqrt{4} = \sqrt{4 \times 36} = \sqrt{144}$

$6 \times 2 = 12$

$\sqrt{49} \times \sqrt{100} = \sqrt{4900}$

$7 \times 10 = 70$

"$\sqrt{49} = \_\_$" asks what number taken as a factor twice gives the product 49?

The number is 7, so we can say

$\sqrt{49} = 7$ $\quad \sqrt{25} = 5$ $\quad \sqrt{100} = 10$

"$\sqrt{49}$" is read, "the square root of 49," and it is another name for the number, 7. We say

$\sqrt{49} = 7$ and $7^2 = 49$

"The square root of 49 is 7," and 7 squared is 49.

If a room is 11 feet wide and 11 feet long, the floor contains 121 square feet.

If the floor of a square room contains 121 square feet, it must be 11 feet by 11 feet.

In the chart below, A stands for the number of square feet in the area of a square floor; and s stands for the number of feet in the length of each side.

	a	b	c	d	e
s	8	10	9	15	12
A	64	100	81	225	144

	f	g	h	i	j
s	21	16	13	20	25
A	441	256	169	400	625

This pattern might be expressed as:

$A = s^2$ or $\sqrt{A} = s$

and it is understood that A refers to the number of square feet in the area of a square, and s refers to the number of feet in the length of each side.

In this sense, we read these expressions as "A equals s squared" and "the square root of A equals s."

But remember that we need to know what A and s refer to, or the expression would not have meaning.

	k	l	m	n	o
	8	10	15	12	9
	128	200	450	288	162

	p	q	r	s	t
	11	14	20	13	22
	242	392	800	338	968

$A = 2s^2$ $\qquad$ $\sqrt{\frac{1}{2}A} = s^2$

Complete the following as true statements:

$\sqrt{9} = 3$ $\quad \sqrt{16} = 4$ $\quad (\frac{1}{2})^2 = \frac{1}{4}$

$4^2 = 16$ $\quad \sqrt{\frac{1}{4}} = \frac{1}{2}$ $\quad 50^2 = 2{,}500$

$\sqrt{.25} = .5$ $\quad 2.5^2 = 6.25$ $\quad \sqrt{625} = 25$

$\sqrt{2{,}500} = 50$ $\quad (\frac{4}{10})^2 = \frac{16}{100}$ or $\frac{4}{25}$ $\quad \sqrt{.09} = .3$

$.5^2 = .25$ $\quad \sqrt{\frac{1}{16}} = \frac{1}{4}$ $\quad \sqrt{6\frac{1}{4}} = 2\frac{1}{2}$

Betty said that she thought the following statements were true. Do you agree?

(1) $\sqrt{8} = \sqrt{4 \cdot 3} = 2 \times \sqrt{2}$ or $2\sqrt{2}$

(2) $\sqrt{45} = \sqrt{9 \cdot 5} = 3\sqrt{5}$

(3) $\sqrt{64} = \sqrt{16 \cdot 4} = 4\sqrt{4}$

Let's test the first example. How much is "the square root of 8?" Or, what is the number of feet in the length of a square that has an area of 8 square feet?

more

less

$2.6 \times 2.6 = 6.76$ $\quad 2.8 \times 2.8 = 7.84$

$2.7 \times 2.7 = 7.29$ $\quad 2.9 \times 2.9 = 8.41$

2.8

2.9

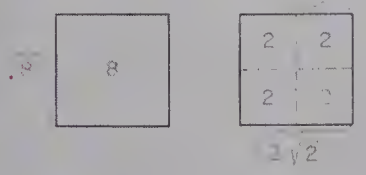
$2.81 \times 2.81 = 7.8961$ $\quad 2.83 \times 2.83 = 8.0089$

$2.82 \times 2.82 = 7.9524$

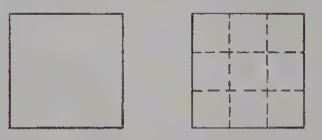
2.83

2.828

What can you say about Betty's first statement . . . that $\sqrt{8} = 2\sqrt{2}$?



Then Jim sat down without saying a word.



Jim came up to the board again. Everyone expected him to draw a sketch to explain Betty's final statement. All Jim wrote was

$\sqrt{64} = 8$ and $4\sqrt{4} = 4 \cdot 2 = 8$

Jim sat down without a word.

Do the sketches and Jim's arithmetic convince you that Betty was right?

Suppose Betty was right. Use the pattern she followed. See if it leads you to any disturbing conclusions.

(Draw more lines if you have a question about the results.)

$\sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$

$\sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$

$\sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2}$

$\sqrt{125} = \sqrt{25 \cdot 5} = 5\sqrt{5}$

$\sqrt{112} = \sqrt{16 \cdot 7} = 4\sqrt{7}$

Indicate the whole number that would make each statement true.

$\sqrt{9} + \sqrt{25} = 8$ $\quad \sqrt{100} \div \sqrt{25} = 2$

$\sqrt{49} \cdot \sqrt{121} = 77$ $\quad \sqrt{3} \cdot \sqrt{3} = 3$

Another pattern:

$$\sqrt{3} \div \sqrt{2} = \sqrt{3 \div 2} = \sqrt{1\tfrac{1}{2}}$$

$$\frac{\sqrt{3}}{\sqrt{2}} = \sqrt{\frac{3}{2}} = \sqrt{1\tfrac{1}{2}}$$

$$\sqrt{36} \div \sqrt{4} = \sqrt{36 \div 4} = \sqrt{9} = 3$$

$$\frac{\sqrt{36}}{\sqrt{4}} = \sqrt{\frac{36}{4}} = \sqrt{9} = 3$$

$$6 \div 2 = 3$$

$$\sqrt{100} \div \sqrt{25} = \sqrt{100 \div 25} = \sqrt{4} = 2$$

$$\frac{\sqrt{100}}{\sqrt{25}} = \sqrt{\frac{100}{25}} = \sqrt{4} = 2$$

$$10 \div 5 = 2$$

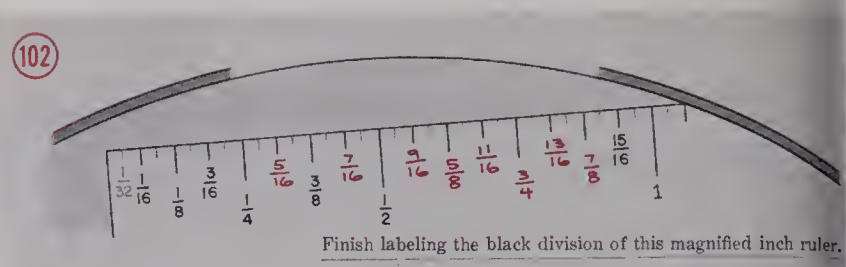
And patterns that turn out to be special cases:

$$\sqrt{4} + \sqrt{4} = 4$$

$$\sqrt{9} + \sqrt{9} \overset{?}{=} 9$$

$$3 + 3 \neq 9$$

After exploring the reliability of patterns, there is a tendency to adopt rules that reflect the pattern. Rules are always dangerous when applied to fresh ideas. Students ought to spend all available time exploring patterns and waste none of it memorizing cues about what is right and wrong. At this point in the development, students ought to try several similar examples when in doubt.



Finish labeling the black division of this magnified inch ruler.

When engineers work in fractions of an inch, they find their computations are easier if they express parts of an inch in decimal fractions rather than common fractions. Since it takes 3 decimal places to find a decimal fraction that is another name for an eighth, they often express all decimal equivalents in thousandths.

They build a table by doing some arithmetic. From then on, they use the table until they have memorized it. Here is the arithmetic. You fill in the answers.

1/8	8) 1.000 → 125	5/8	8) 5.000 → .625
1/4	4) 1.000 → .250	3/4	4) 3.000 → 750
3/8	8) 3.000 → .375	7/8	8) 7.000 → 875
1/2	2) 1.000 → 500	1/1	1) 1.000 → 1.000

To express sixteenths as decimal equivalents, a fourth decimal place is used.

1/16	16) 1.0000 → 0625	9/16	16) 9.0000 → 5625
3/16	16) 3.0000 → .1875	11/16	16) 11.0000 → .6875
5/16	16) 5.0000 → .3125	13/16	16) 13.0000 → .8125
7/16	16) 7.0000 → .4375	15/16	16) 15.0000 → .9375

An engineer wants to have six holes drilled in a straight line, with centers $1\frac{1}{16}$ inches apart. What is the distance between the centers of the two holes that are furthest apart?

7.1875 in. (5 × 1.4375)

When the engineer measured the finished piece, he found that the end holes were .007 of an inch further apart than he had requested. They were 7.1945 apart.

How might an engineer express the following as decimal fractions?

$1\frac{1}{16}$" → 1.0625"	$7\frac{11}{16}$" → 7.6875"
$12\frac{3}{8}$" → 12.375"	$4\frac{1}{4}$" → 4.250"
$3\frac{1}{5}$" → 3.200"	$17\frac{3}{5}$" → 17.600"
$\frac{1}{20}$" → .05"	$\frac{7}{25}$" → .280"
$\frac{17}{50}$" → .340"	$1\frac{3}{20}$" → 1.150"

Engineers often need to work in thirty-seconds and sixty-fourths of an inch. Change the following common fractions to decimal fractions:

1/32 = .03125	1/64 = .015625
6/32 = .1875	7/64 = .109375
11/32 = .34375	40/64 = .625
28/32 = .875	19/64 = .296875

102

1/64 .015625
1/32 .03125
3/64 .046875
1/16 .0625
5/64 .078125
3/32 .09375
7/64 .109375
1/8 .125
9/62 .140625
5/32 .15625
11/64 .171875
3/16 .1875
13/64 .203125
7/32 .21875
15/64 .234375
1/4 .250
1/2 .500
1 1.000

Notes to accompany *Work Text* page 102

Engineers and technicians and draftsmen use conversion tables to remind them of the results we reach on page 102. Perhaps the culmination of the work on this page might be the preparation of such a table for future use by the students. It starts out like this:

Notes to accompany *Work Text* page 103

Banks and other institutions use interest tables instead of repeated computations. Each of several groups in the class might be asked to prepare an interest table for a given rate of interest for three month periods of time, such as the one below:

RATE: 3 1/2%

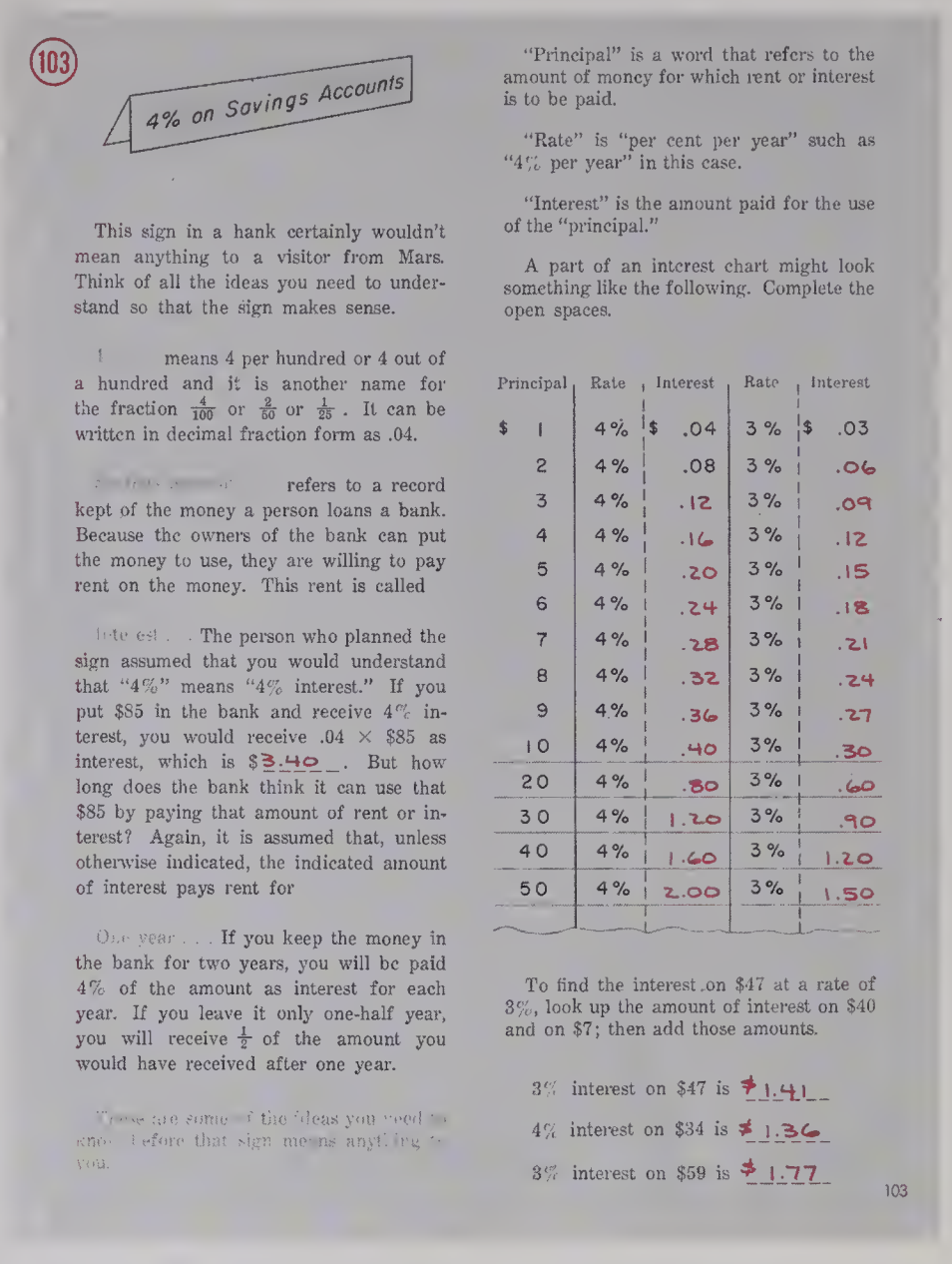
	3 mon.	6 mon.	9 mon.	1 year
$ 1	.01	.02	.03	.04
2	.02	.04	.06	.07
3	.03	.06	.09	.11
4	.04	.07	.11	.14
5	.05	.09	.14	.18
6	.05	.11	.16	.21
7	.06	.13	.19	.25
8	.07	.14	.21	.28
9	.08	.16	.24	.32
10	.09	.18	.27	.35
20	.18	.35	.53	.70
30	.26	.53	.79	1.05
40	.35	.70	1.05	1.40
50	.44	.88	1.32	1.75
60	.53	1.05	1.58	2.10
70	.62	1.23	1.85	2.45
80	.70	1.40	2.10	2.80
90	.79	1.58	2.37	3.15
100	.88	1.75	2.63	3.50
200	1.75	3.50	5.25	7.00
300	2.63	5.25	7.88	10.50
400	3.50	7.00	10.50	14.00
500	4.38	8.75	13.13	17.50
600	5.25	10.50	15.75	21.00
700	6.13	12.25	18.38	24.50
800	7.00	14.00	21.00	28.00
900	7.88	15.75	23.63	31.50
1,000	8.75	17.50	26.25	35.00

The ability to read such charts is as important as understanding how they are constructed. So, after the charts are developed, provide opportunities to use them, perhaps by asking for completion of headlines and appropriate stories:

3 1/2% of 758 = ____, and
3/4 × 3 1/2% × 1,825 = ____

Notes to accompany *Work Text* page 104

Page 104 returns to the recurrent theme of "numbers have many names."



103

4% on Savings Accounts

This sign in a bank certainly wouldn't mean anything to a visitor from Mars. Think of all the ideas you need to understand so that the sign makes sense.

4 means 4 per hundred or 4 out of a hundred and it is another name for the fraction $\frac{4}{100}$ or $\frac{2}{50}$ or $\frac{1}{25}$. It can be written in decimal fraction form as .04.

[illegible] refers to a record kept of the money a person loans a bank. Because the owners of the bank can put the money to use, they are willing to pay rent on the money. This rent is called

Interest. The person who planned the sign assumed that you would understand that "4%" means "4% interest." If you put $85 in the bank and receive 4% interest, you would receive .04 × $85 as interest, which is $3.40. But how long does the bank think it can use that $85 by paying that amount of rent or interest? Again, it is assumed that, unless otherwise indicated, the indicated amount of interest pays rent for

One year. . . If you keep the money in the bank for two years, you will be paid 4% of the amount as interest for each year. If you leave it only one-half year, you will receive $\frac{1}{2}$ of the amount you would have received after one year.

These are some of the ideas you need to know before that sign means anything to you.

"Principal" is a word that refers to the amount of money for which rent or interest is to be paid.

"Rate" is "per cent per year" such as "4% per year" in this case.

"Interest" is the amount paid for the use of the "principal."

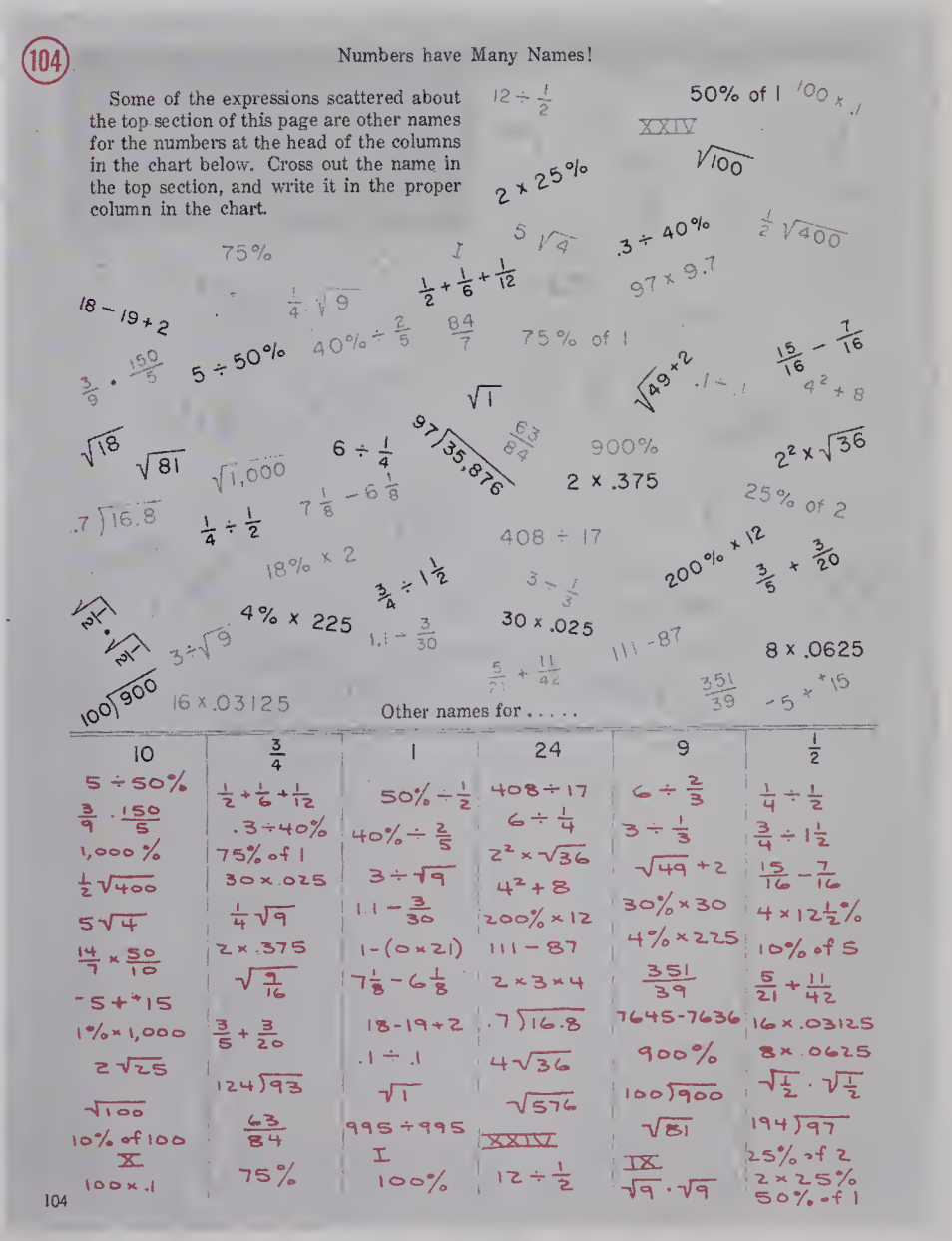
A part of an interest chart might look something like the following. Complete the open spaces.

Principal	Rate	Interest	Rate	Interest
$ 1	4%	$ .04	3%	$ .03
2	4%	.08	3%	.06
3	4%	.12	3%	.09
4	4%	.16	3%	.12
5	4%	.20	3%	.15
6	4%	.24	3%	.18
7	4%	.28	3%	.21
8	4%	.32	3%	.24
9	4%	.36	3%	.27
10	4%	.40	3%	.30
20	4%	.80	3%	.60
30	4%	1.20	3%	.90
40	4%	1.60	3%	1.20
50	4%	2.00	3%	1.50

To find the interest on $47 at a rate of 3%, look up the amount of interest on $40 and on $7; then add those amounts.

3% interest on $47 is $1.41
4% interest on $34 is $1.36
3% interest on $59 is $1.77

103



104

Numbers have Many Names!

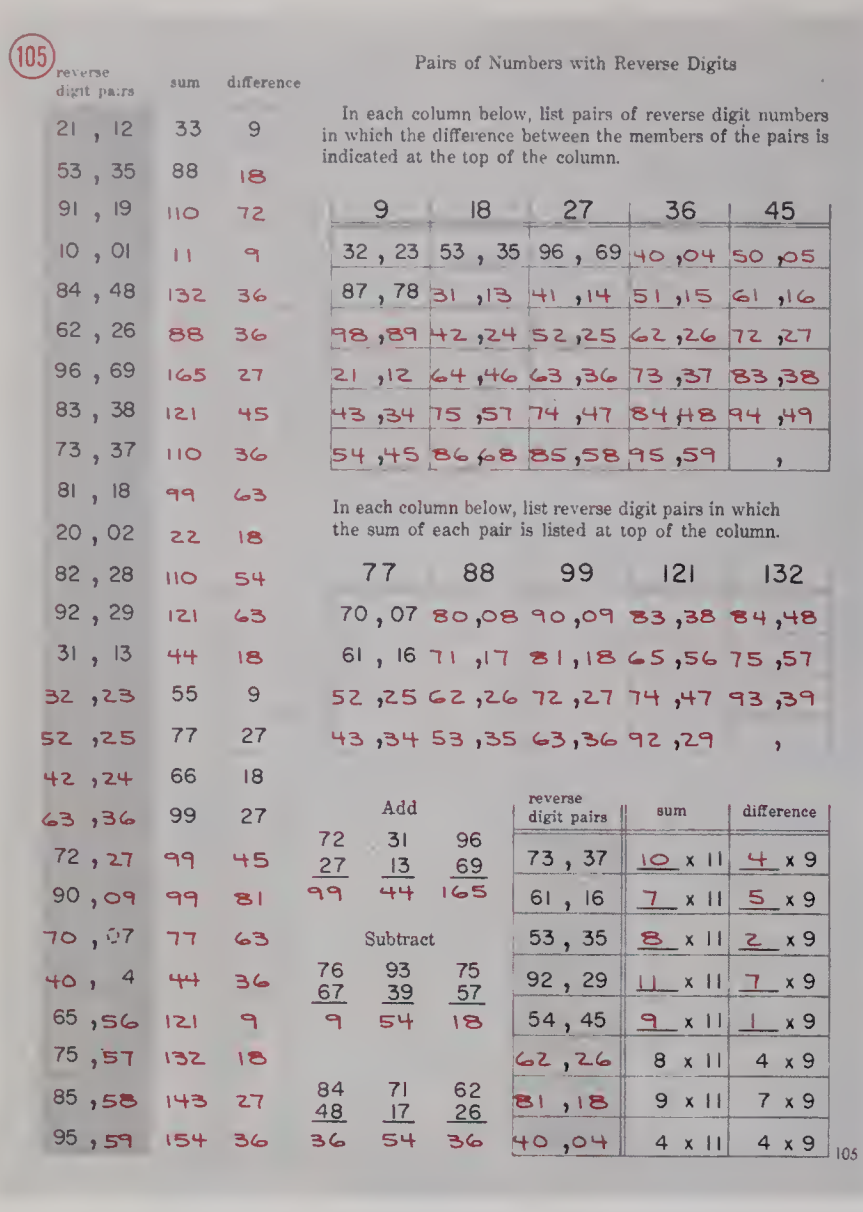
Some of the expressions scattered about the top section of this page are other names for the numbers at the head of the columns in the chart below. Cross out the name in the top section, and write it in the proper column in the chart.

12 ÷ 1/2 · 50% of 1 · 100 × .1 · XXIV · √100 · 2 × 25% · 5√4 · .3 ÷ 40% · 1/2 √400 · 75% · 1/2 + 1/6 + 1/12 · 97 × 9.7 · 1/4 · √9 · 18 − 19 + 2 · 84/7 · 3/9 · 150/5 · 5 ÷ 50% · 40% ÷ 2/5 · 75% of 1 · √1 · √49 + 2 · .1 ÷ .1 · 15/16 − 7/16 · 4² + 8 · √18 · √81 · √1,000 · 97 ⟌ 35,876 · 63/84 · 900% · 2 × .375 · 2² × √36 · 6 ÷ 1/4 · .7 ⟌ 16.8 · 7 1/8 − 6 1/8 · 25% of 2 · 1/4 ÷ 1/2 · 408 ÷ 17 · 18% × 2 · 3/4 ÷ 1 1/2 · 200% × 12 · 3/5 + 3/20 · 3 − 1/3 · 4% × 225 · 30 × .025 · 111 − 87 · 8 × .0625 · 3 ÷ √9 · 1.1 − 3/30 · 5/21 + 11/42 · 351/39 · −5 × −15 · 100 ⟌ 900 · 16 × .03125

Other names for . . . . .

10	3/4	1	24	9	1/2
5 ÷ 50%	1/2 + 1/6 + 1/12	50% ÷ 1/2	408 ÷ 17	6 ÷ 2/3	1/4 ÷ 1/2
3/9 · 150/5	.3 ÷ 40%	40% ÷ 2/5	6 ÷ 1/4	3 ÷ 1/3	3/4 ÷ 1 1/2
1,000 %	75% of 1	3 ÷ √9	2² × √36	√49 + 2	15/16 − 7/16
1/2 √400	30 × .025	1.1 − 3/30	4² + 8	30% × 30	4 × 12 1/2 %
5√4	1/4 √9	1 − (0 × 21)	200% × 12	4% × 225	10% of 5
14/7 × 50/10	2 × .375	7 1/8 − 6 1/8	111 − 87	351/39	5/21 + 11/42
−5 + −15	√3/16	18 − 19 + 2	2 × 3 × 4	7645 − 7636	16 × .03125
1% × 1,000	3/5 + 3/20	.1 ÷ .1	.7 ⟌ 16.8	900%	8 × .0625
2√25	124 ⟌ 93	√1	4√36	100 ⟌ 900	√1/2 · √1/2
√100	63/84	995 ÷ 995	√576	√81	194 ⟌ 97
10% of 100	75%	I	XXIV	IX	25% of 2
X		100%	12 ÷ 1/2	√9 · √9	2 × 25%
100 × .1					50% of 1

104



105

Pairs of Numbers with Reverse Digits

reverse digit pairs	sum	difference
21 , 12	33	9
53 , 35	88	18
91 , 19	110	72
10 , 01	11	9
84 , 48	132	36
62 , 26	88	36
96 , 69	165	27
83 , 38	121	45
73 , 37	110	36
81 , 18	99	63
20 , 02	22	18
82 , 28	110	54
92 , 29	121	63
31 , 13	44	18
32 , 23	55	9
52 , 25	77	27
42 , 24	66	18
63 , 36	99	27
72 , 27	99	45
90 , 09	99	81
70 , 07	77	63
40 , 4	44	36
65 , 56	121	9
75 , 57	132	18
85 , 58	143	27
95 , 59	154	36

In each column below, list pairs of reverse digit numbers in which the difference between the members of the pairs is indicated at the top of the column.

9	18	27	36	45
32 , 23	53 , 35	96 , 69	40 , 04	50 , 05
87 , 78	31 , 13	41 , 14	51 , 15	61 , 16
98 , 89	42 , 24	52 , 25	62 , 26	72 , 27
21 , 12	64 , 46	63 , 36	73 , 37	83 , 38
43 , 34	75 , 57	74 , 47	84 , 48	94 , 49
54 , 45	86 , 68	85 , 58	95 , 59	,

In each column below, list reverse digit pairs in which the sum of each pair is listed at top of the column.

77	88	99	121	132
70 , 07	80 , 08	90 , 09	83 , 38	84 , 48
61 , 16	71 , 17	81 , 18	65 , 56	75 , 57
52 , 25	62 , 26	72 , 27	74 , 47	93 , 39
43 , 34	53 , 35	63 , 36	92 , 29	,

Add

72	31	96
27	13	69
99	44	165

Subtract

76	93	75
67	39	57
9	54	18

84	71	62
48	17	26
36	54	36

reverse digit pairs	sum	difference
73 , 37	10 x 11	4 x 9
61 , 16	7 x 11	5 x 9
53 , 35	8 x 11	2 x 9
92 , 29	11 x 11	7 x 9
54 , 45	9 x 11	1 x 9
62 , 26	8 x 11	4 x 9
81 , 18	9 x 11	7 x 9
40 , 04	4 x 11	4 x 9

105

about all number listed under 'sum?'" Answer: all are multiples of 11.

"What do you notice about all numbers listed under 'difference?'" Answer: all are multiples of 9.

"Do these two patterns apply to all pairs of 2-digit numbers that meet the reverse-digit condition?" Yes. "Why?"

Consider several pairs: 21,12 and 53,35 and 96,69. The sums and differences can be written in this way:

21,12	$2 \times 10 + 1$	$2 \times 10 + 1$
	$+ 1 \times 10 + 2$	$- 1 \times 10 - 2$
	$3 \times 10 + 3$	$1 \times 10 - 1$
	$3 (10 + 1)$	$1 (10 - 1)$
	$3 (11)$	$1 (9)$
53,35	$5 \times 10 + 3$	$5 \times 10 + 3$
	$+ 3 \times 10 + 5$	$- 3 \times 10 - 5$
	$8 \times 10 + 8$	$2 \times 10 - 2$
	$8 (10 + 1)$	$2 (10 - 1)$
	$8 (11)$	$2 (9)$
96,69	$9 \times 10 + 6$	$9 \times 10 + 6$
	$+ 6 \times 10 + 9$	$- 6 \times 10 - 9$
	$15 \times 10 + 15$	$3 \times 10 - 3$
	$15 (10+1)$	$3 (10-1)$
	$15 (11)$	$3 (9)$

In general:

$a \times 10 + b$	$a \times 10 + b$
$b \times 10 + a$	$- b \times 10 - a$
$(a+b) \cdot 10 + (a+b)$	$(a-b)\ 10 + (b-a)$
$(a+b)\ (10+1)$	$(a-b)\ 10 - (a-b)$
$(a+b)\ (11)$	$(a-b)\ (10-1)$
	$(a-b)\ (9)$

In the chart at the top on the right, a series of patterns develops:

- —under 9 in all examples, the difference between the digits is 1
- —under 18 in all examples, the difference between the digits is 2
- —under 27 in all examples, the difference between the digits is 3
- —under *4* × 9 in all examples, the difference between the digits is 4
- —under *5* × 9 in all examples, the difference between the digits is 5

In the next chart:

- —under 11 × *12*, the sum of the digits in each example is 12
- —under 11 × *9*, the sum of the digits in each example is 9

$$\begin{array}{r} 13 \\ +\ 89 \\ \hline 73 \\ -\ 58 \\ \hline 59 \\ \times\ \ 8 \\ \hline \end{array} \quad \begin{array}{l} \rightarrow \\ \\ \rightarrow \\ \\ \rightarrow \\ \\ \end{array} \quad \begin{array}{l} 10 + 3 \\ 80 + 9 \\ \hline 60 + 13 \\ -\ (50 + \ \ 8) \\ \hline 50 + 9 \\ \times\ \ 8 \\ \hline \end{array}$$

$\overline{12)168} \rightarrow \overline{12)120} + \overline{12)48}$

$4/12 \rightarrow 1/3$, or $1/3 \rightarrow 4/12$ (as in $1/3 + 1/4 = 4/12 + 3/12 = 7/12$

$3\ 1/2\% \rightarrow .035$

$\sqrt{128} \rightarrow 8\sqrt{2}$

Discuss the role of "other names for numbers" in each of these examples. Ask students to prepare a very complex looking name for each of the six numbers considered at the bottom of *Work Text* page 104, such as:

$$\frac{1/3 + \sqrt{4/9}}{2(8{,}328 \div 16{,}656} = \underline{\quad}$$ (Answer: 1)

Notes to accompany *Work Text* page 105

Page 105 is a pattern hunt that provides a change of pace. "In the left-hand chart, what do you notice

The two numbers −11 and 9−occur again and again. Our system of numeration is based on 10. The two numbers 11 and 9 can be expressed as "base + 1" and "base − 1". (Those students who are familiar with other bases might like to find out whether this pattern occurs in systems of numeration with other bases. They will find, of course, that it does.)

Notes to accompany *Work Text* pages 106-107

Pages 106 and 107 reveal patterns that are most useful shortcuts in certain computational situations.

"Think of a number more than 5. Add 5 to the number you thought of and write down the result. Next, subtract 5 from the number you thought of and write down the result. Multiply the two numbers you have noted and tell me your result."

The answer might be "200."

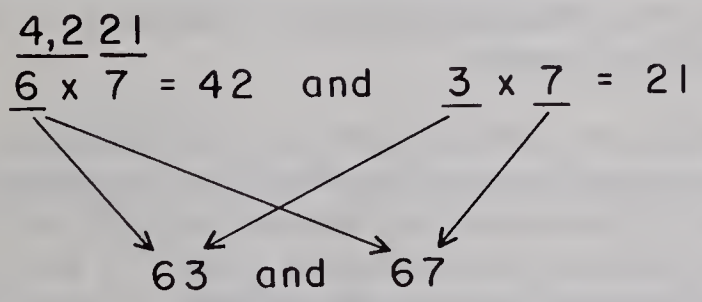
"The number you thought of must have been 15." . . .and you are a mind reader in the eyes of some. Others know your argument:

$$200 + 5^2 = 225 \text{ and } \sqrt{225} = 15$$

Another−"From any decade, select two numbers so that the sum of the digits in the units place is 10. Multiply these two numbers and tell me your result."

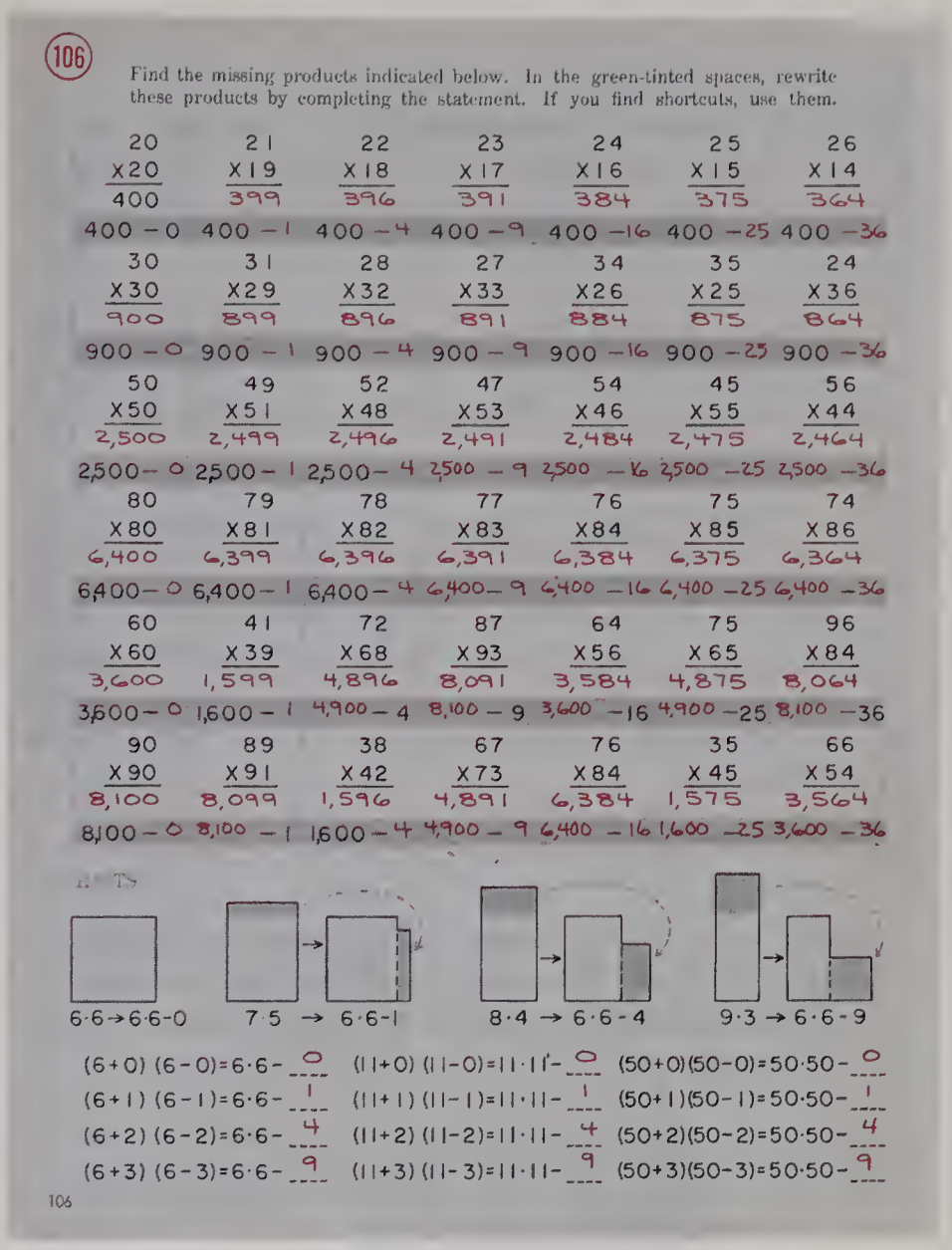
The answer might be "4,221."

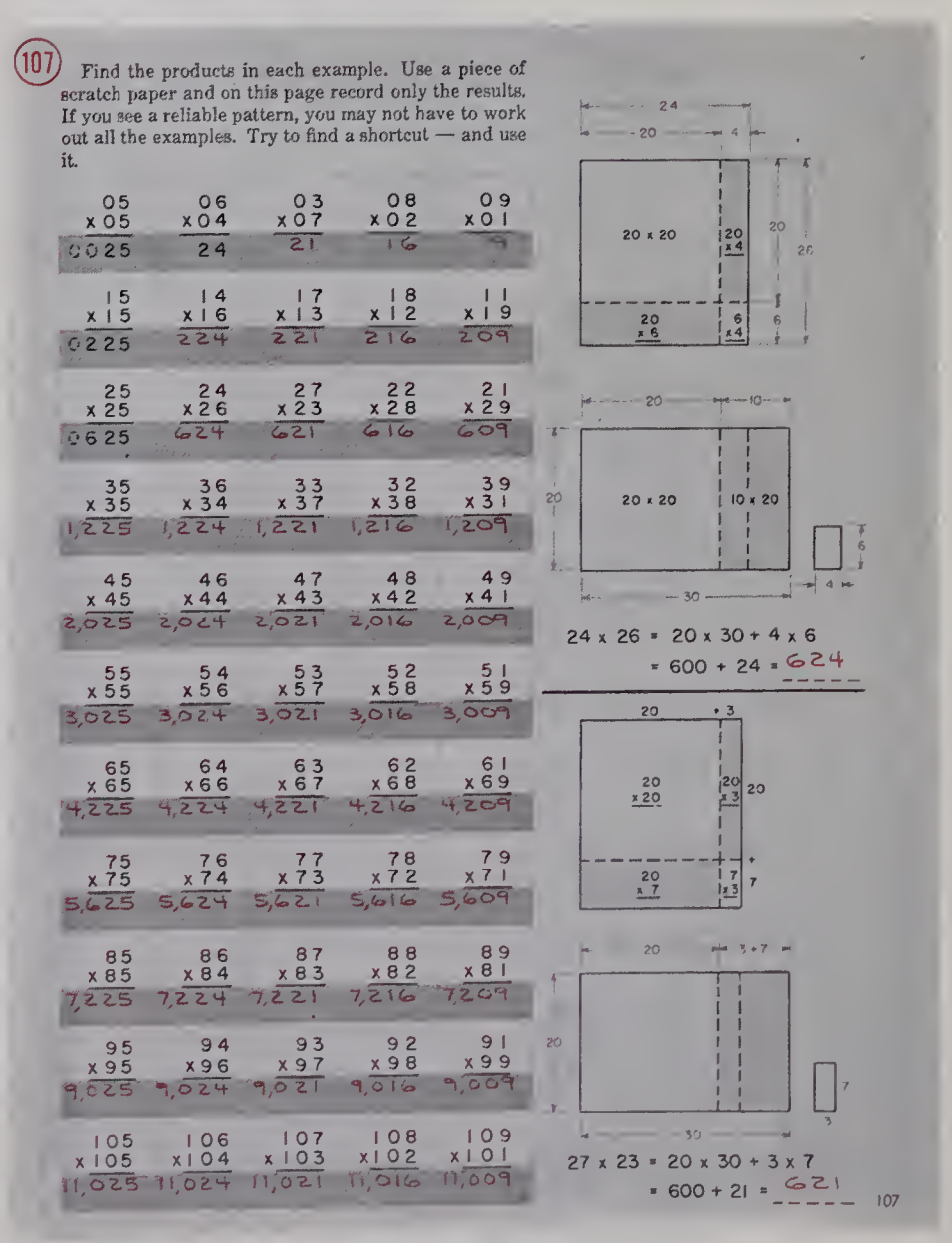
"You numbers were 63 and 67." . . .and you are a mind reader again, unless someone knows you argued:



Anyone familiar with the developments on these pages 106 and 107 can perform these "mind reading" feats. After a few introductory examples, let students work independently through these two pages. Your introduction will have motivated a search for patterns as the work proceeds.

Postpone discussion until everyone has finished the work. And before any discussion, change your introductory tactics to an approach like this:





"I thought of a number. To this number I added 4, and to the number I thought of I subtracted 4. I multiplied these results. The product is 384. What number did I think of?"

And, "From a decade, I selected a pair of numbers. The sum of the digits in the units place was 10. I multiplied them. The product is 7,216. What number did I select?"

By this time, students can take over, reading minds and posing questions. Then let the group discuss the patterns on pages 106 and 107. Draw sketches on the board to help clarify questions. Do not depend on computations alone. You may want to generalize the idea on page 106 with this pattern indicator:

$$(a+b)\,(a-b) = a^2 - b^2$$

The pattern on page 107 is more complex in its general form. You may want to omit any discussion of the following or limit the discussion to a few students:

$$\begin{array}{r} 100 + 10 - b \\ \times\ 100 + b \\ \hline 100\,a^2 + 100\,a - 10\,ab \\ \qquad\qquad + 10\,ab + 10\,b - b^2 \\ \hline 100\,a^2 + 100\,a \qquad\quad + 10b - b^2 \end{array}$$

$$100\,a\,(a+1) \quad + b\,(10 - b)$$

Example:

$$\begin{array}{r} 43 \\ \times\ 47 \\ \hline \end{array}$$

$$\begin{array}{r} (10\cdot4) \quad + \quad (10-7) \\ \times\ (10\cdot4) \quad + \ 7 \\ \hline 100\cdot4^2 \quad + \quad 100\cdot4 \ - \ 100\cdot4\cdot7 \\ \qquad\qquad\qquad\quad + \ 7\cdot10\cdot4 + 7\cdot10 \ - \ 7\cdot7 \\ \hline 100\cdot4^2 + 100\cdot4 \qquad + 7\cdot10 \ - \ 7\cdot7 \end{array}$$

$$100\cdot4\ (4+1) + 7\,(10-7)$$

$$100\ (4\cdot5) + \underline{7\cdot3} = \underline{2}000 + \underline{21} = \underline{20}\underline{21}$$

unless general development could be considered:

$$\begin{array}{r} 35 \\ \times\ 35 \\ \hline 1225 \end{array}$$

$$\begin{array}{r} 10a \ + 5 \\ 10a \ + 5 \\ \hline 100a^2 + 50a \\ \qquad\quad 50a + 25 \\ \hline 100a^2 + 100a + 25 \end{array} = 100a\,(a+1) + 25$$

However, all such extensions are optional.

Notes to accompany *Work Text* pages 108-128

The final series of pages in *Work Text Level F* is designed as a summary of the basic patterns of arithmetic—a generalization of arithmetic or a foray into algebra.

Many of the important generalizations we want to talk about are so obvious to students they aren't sure we are serious in broaching them. For example, we notice that:

$3 + 4 = 4 + 3$ $\qquad$ $\sqrt{4} + 7 = 7 + \sqrt{4}$
$9 + 2 = 2 + 9$ $\qquad$ $1/2 + .5 = .5 + 1/2$

we use a "pattern indicator"

$$a + b = b + a$$

and we dignify this idea with a high-sounding name: "the commutative principle for addition."

Students need a fresh way of looking at such seemingly obvious ideas. Beginning on *Work Text* page 108, we employ the device of a visitor from Mars establishing space-phone contact with Earth, and asking, "What is arithmetic all about?"

Where do you begin?

We must explain the obvious, because it is not obvious to the space ship caller. He does not even know what a *number* is, nor the meaning of *addition.* To him, our pattern indicators such as a + b = b + a are senseless sounds.

We develop a plan of attack. We despair of explaining the endless details we have learned over the years. We decide, rather, to talk about the basic patterns so our man from Mars can work out the details for himself.

Pages 108, 109 and 110 begin the explanation of some elementary patterns. Page 110 recognizes the need for an overall plan of attack and outlines the main features of such a plan. Pages 111 and 112 (and the top of page 113) discuss each of the aspects of this plan briefly, and introduces some of the mathematical terminology.

Page 113 discusses the question of whether subtraction is always possible and leads to a consideration of negative and positive numbers on pages 114 and 115.

Pages 116 through 119 consider the question of whether or not division is always possible—and, it

is, of course, when we introduce fractions, including a fraction as divisor, but excluding 0 as a divisor.

Page 120 reviews the fundamental patterns on which arithmetic is based. Pages 121 through 125 look more deeply into the four questions:

a) Is addition always possible?
b) Is subtraction always possible?
c) Is multiplication always possible?
d) Is division always possible?

Pages 126, 127 and 128 are a novel approach to the uses of arithmetic to solve, in the same basic pattern, measurement problems on Mars and on earth.

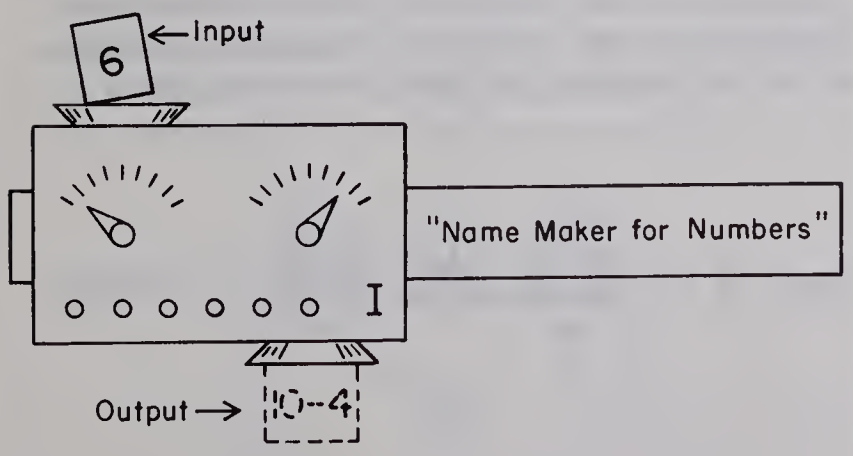
The entire last section is a study of the generalizations of arithmetic which are the ideas upon which algebra is based. The emphasis is not on manipulations – it is focused on understanding basic concepts and, after that understanding has taken root, secondary concern is with the naming of those concepts.

Read through *Work Text* pages 108 to 128 before you help the class work this section.

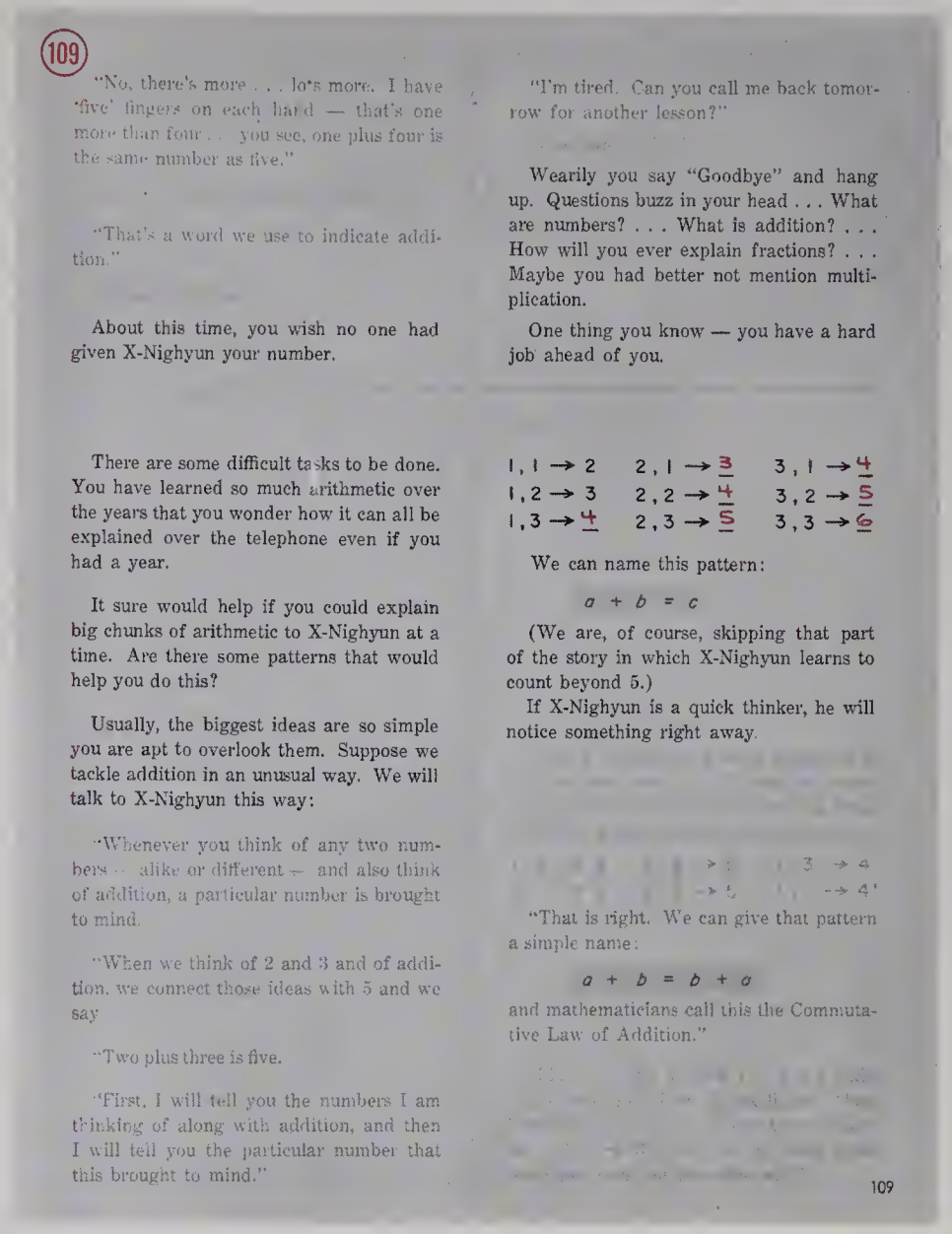
A parallel turnpike

Consider developing the suggestions that follow in one of three ways: (1) as readiness for this section; (2) as material to be developed in parallel with this section; or, (3) as a final reinforcement of the generalizations of arithmetic.

Let us create a character to be known as "Machine Mike." He invents and builds machines that are based on the basic patterns of arithmetic. His first machine is a "Name Maker for Numbers." You write a number on a piece of paper and feed the paper into the "Name Maker" – Machine I.



Write 6 on a piece of paper, set the dial, and out comes a piece of paper on which is written 10-4. Change the dials - out comes 2 × 3. Change again,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"No, there's more . . . lots more. I have 'five' fingers on each hand — that's one more than four . . you see, one plus four is the same number as five."

"That's a word we use to indicate addition."

About this time, you wish no one had given X-Nighyun your number.

"I'm tired. Can you call me back tomorrow for another lesson?"

Wearily you say "Goodbye" and hang up. Questions buzz in your head . . . What are numbers? . . . What is addition? . . . How will you ever explain fractions? . . . Maybe you had better not mention multiplication.

One thing you know — you have a hard job ahead of you.

There are some difficult tasks to be done. You have learned so much arithmetic over the years that you wonder how it can all be explained over the telephone even if you had a year.

It sure would help if you could explain big chunks of arithmetic to X-Nighyun at a time. Are there some patterns that would help you do this?

Usually, the biggest ideas are so simple you are apt to overlook them. Suppose we tackle addition in an unusual way. We will talk to X-Nighyun this way:

"Whenever you think of any two numbers — alike or different — and also think of addition, a particular number is brought to mind.

"When we think of 2 and 3 and of addition, we connect those ideas with 5 and we say

"Two plus three is five.

"First, I will tell you the numbers I am thinking of along with addition, and then I will tell you the particular number that this brought to mind."

1, 1 → 2 2, 1 → 3 3, 1 → 4
1, 2 → 3 2, 2 → 4 3, 2 → 5
1, 3 → 4 2, 3 → 5 3, 3 → 6

We can name this pattern:

$a + b = c$

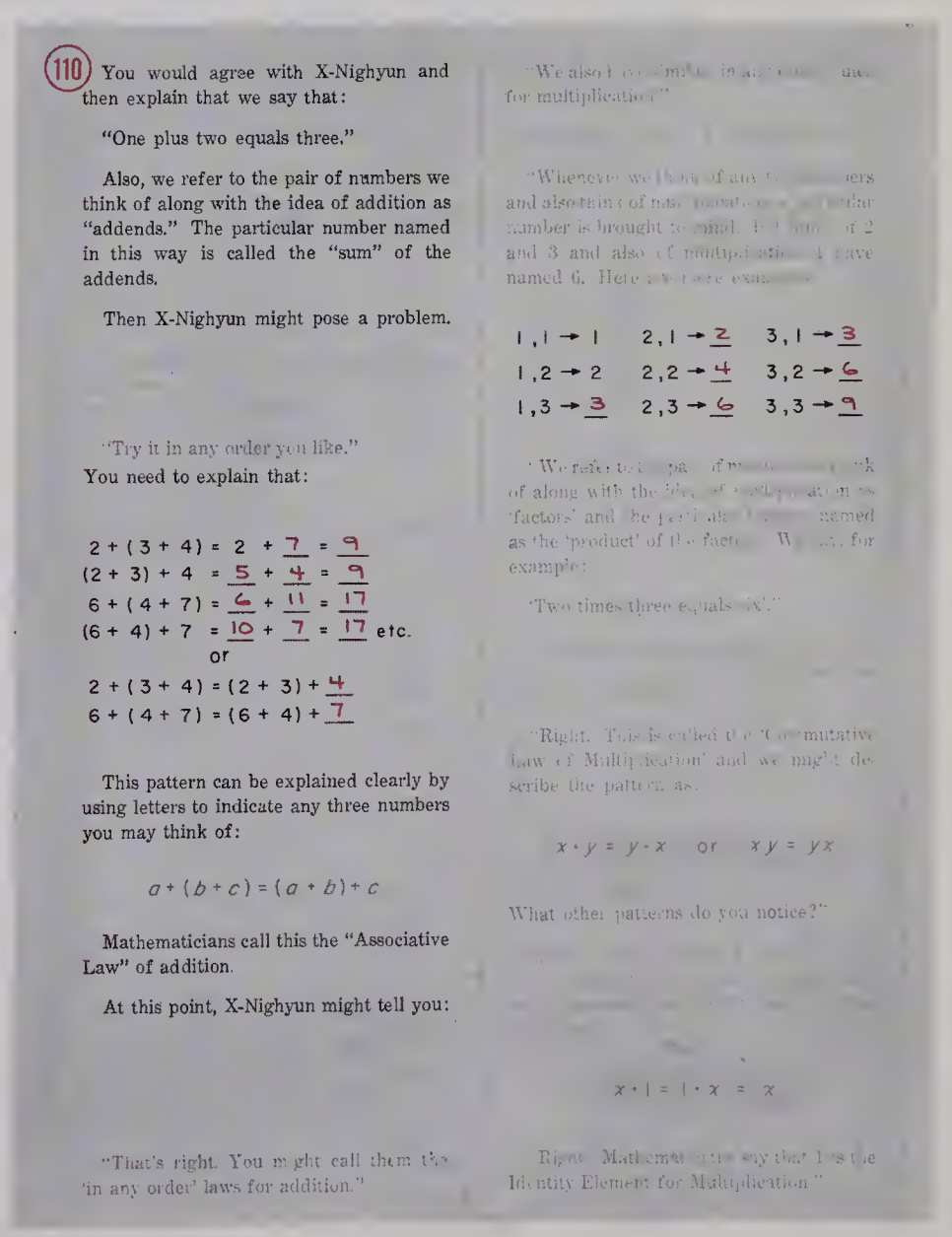
(We are, of course, skipping that part of the story in which X-Nighyun learns to count beyond 5.)

If X-Nighyun is a quick thinker, he will notice something right away.

"That is right. We can give that pattern a simple name:

$a + b = b + a$

and mathematicians call this the Commutative Law of Addition."


110

You would agree with X-Nighyun and then explain that we say that:

"One plus two equals three."

Also, we refer to the pair of numbers we think of along with the idea of addition as "addends." The particular number named in this way is called the "sum" of the addends.

Then X-Nighyun might pose a problem.

"Try it in any order you like."

You need to explain that:

2 + (3 + 4) = 2 + 7 = 9
(2 + 3) + 4 = 5 + 4 = 9
6 + (4 + 7) = 6 + 11 = 17
(6 + 4) + 7 = 10 + 7 = 17 etc.
or
2 + (3 + 4) = (2 + 3) + 4
6 + (4 + 7) = (6 + 4) + 7

This pattern can be explained clearly by using letters to indicate any three numbers you may think of:

$a + (b + c) = (a + b) + c$

Mathematicians call this the "Associative Law" of addition.

At this point, X-Nighyun might tell you:

"That's right. You might call them the 'in any order' laws for addition."

"We also [illegible] for multiplication."

"Whenever we [illegible] and also think of [illegible] number is brought to mind. [illegible] of 2 and 3 and also of multiplication [illegible] named 6. Here are more examples:

1, 1 → 1 2, 1 → 2 3, 1 → 3
1, 2 → 2 2, 2 → 4 3, 2 → 6
1, 3 → 3 2, 3 → 6 3, 3 → 9

"We refer to the pair of numbers we think of along with the idea of multiplication as 'factors' and the particular number named as the 'product' of the factors. We say, for example:

'Two times three equals six.'

"Right. This is called the 'Commutative Law of Multiplication' and we might describe the pattern as:

$x \cdot y = y \cdot x$ or $xy = yx$

What other patterns do you notice?"

$x \cdot 1 = 1 \cdot x = x$

Right. Mathematicians say that 1 is the Identity Element for Multiplication."

"You are right again."

$2 \times (3 \times 5) = (2 \times 3) \times 5$?"

"Now find out for yourself whether the Associative Law applies to multiplication. Is it true that

"Arrange the books in two columns and three columns. Now call these 5 layers. Put 4 more layers on top — 5 layers in all. How many books?"

"You'll have to work that problem out for yourself."

"Right. Seven and five more, for a total of twelve. I was thinking 4 and 3 and the idea of multiplication, we say that product of 3 and 4 is 12."

"Well, [illegible]"

$a \cdot (b + c) = a \cdot b + a \cdot c$

$2(3+4) = 2 \cdot 7 = 14$
$2(3+4) = 2 \cdot 3 + 2 \cdot 4 = \underline{6} + \underline{8} = \underline{14}$

$7(8+2) = 7 \cdot 10 = \underline{70}$
$7(8+2) = 7 \cdot 8 + \underline{7 \cdot 2} = \underline{56} + \underline{14} = \underline{70}$

"We use this [illegible] multiplication. We [illegible] product of 7 and 86. We [illegible] of its other names — 80 + 6 — and we say

$7(80+6) = 7 \cdot 80 + 7 \cdot 6 = \underline{560} + \underline{42} = \underline{602}$

It will also be necessary to explain that we can find a missing addend when one addend and the sum are given. If a is any sum and b is a given addend, we can find the other addend x.

$a - b = x$

Also, if a is any given product and b is a known factor, we can find the missing factor x.

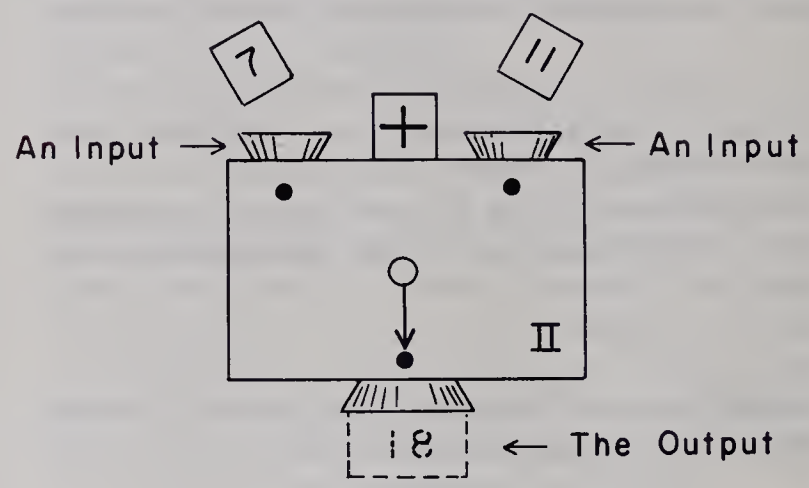
$a \div b = x$

We will explain that finding a missing addend is called "subtraction" and that finding a missing factor is called "division."

Another that is helpful when division examples are considered, shows:

Input	→	Output	
144		120 + 24	(12)144)
92		80 + 12	(4)92)
4,620		3,500 + 1,050 + 70	(35)4,620)
.72		.60 + .12	(3).72)

All other machines have a different kind of input. Each requires a piece of paper with a number written on it. There is Machine II. It looks something like the sketch below, in which the sample inputs are 7 and 11.



and out comes $\sqrt{36}$; again, out comes $2\sqrt{9}$; again, $324 \div 54$.

Since 7 + 2 is a number, you may write 7 + 2 on a piece of paper and feed it into the "Number Maker"—and, as you change dial settings, out come such names as:

3×3	$2 \times 2 \times 2 + 1$
$2(18 \div 4)$	$7.329 + 1.671$
$\sqrt{81}$	$4 + 3/4 + 4\ 1/4$
900%	$9{,}000 \div 1{,}000$

One particular setting shows the following results:

Input	→	Output
12		10 + 2
83		80 + 3
129		100 + 20 + 9
1.8		1 + .8
7.19		7 + .1 + .09

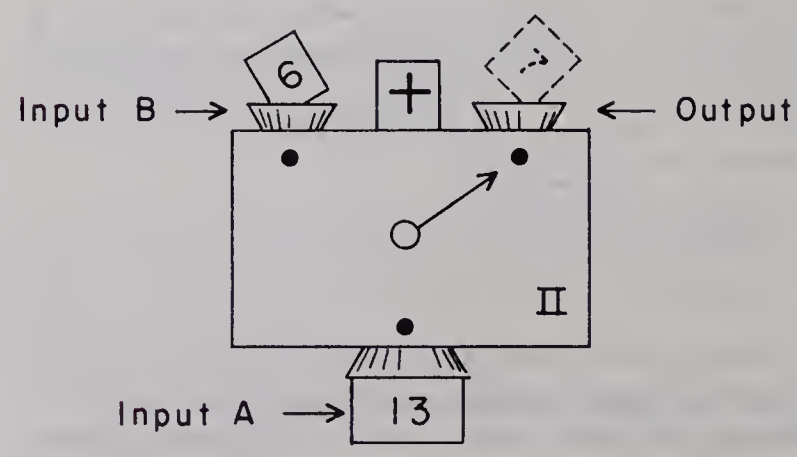
Another setting (of use in subtraction) shows these results:

Input	→	Output
39		20 + 19
127		110 + 17
724		600 + 110 + 14

Typical of its results are:

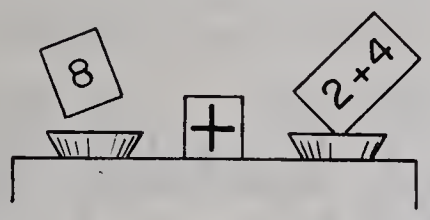
Inputs			Output
7 ,	11	⟶	18
3 ,	98	⟶	103
1/2 ,	1/3	⟶	5/6
.9 ,	1.4	⟶	2.3
$\sqrt{16}$,	158.3	⟶	162.3
$^{+}3$,	$^{-}5$	⟶	$^{-}2$
10% ,	17%	⟶	27%

This machine has a dial that points to the output—and the dial can designate any of the three openings as the output. When the inputs are as shown below, the following are some possible results:



Inputs A	B		Output
124 ,	38	⟶	86
1/2 ,	1/3	⟶	1/6
.8 ,	.17	⟶	.63
√100 ,	√64	⟶	2
⁻8 ,	⁻3	⟶	⁻5

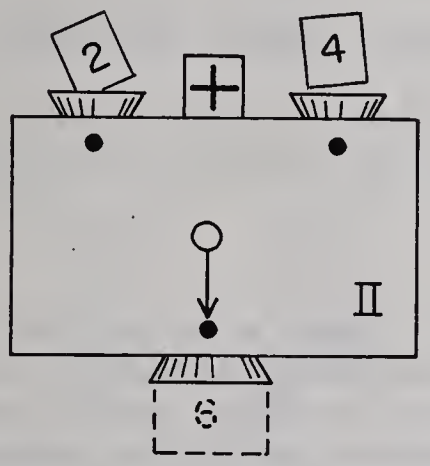
Machine II will not accept this kind of input:



It will not act as a computer in the situation:

8 + (2+4)

However, it is still useful, because a preliminary step shows us:

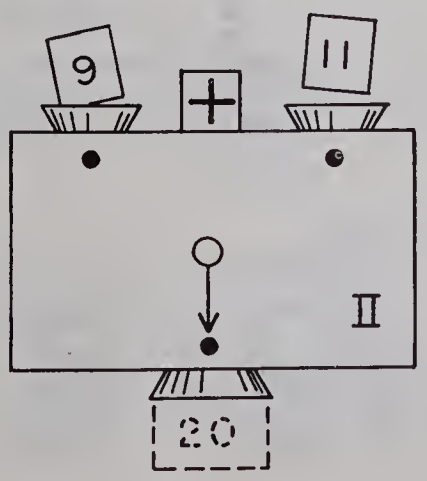


So, the situation has now been transformed by the machine from:

8 + (2+4) into 8 + 6

which the machine is designed to handle. With 8 and 6 as inputs, and the arrow pointing down, the output is 14.

Confronted with (2+7) + (8+3), the operator must make two preliminary uses of Machine I. The first indicates that 2 + 7 = 9; the second, that 8 + 3 = 11. Now, the machine can handle the situation:



(113) Finally, we will explain that if the product of a pair of factors is zero, then one or both of the factors must be zero.

If $a \cdot b = 0$, then $a = 0$ or $b = 0$

X-Nighyun Raises Two Questions

X-Nighyun gave this set of examples:

2 - 0 = 2	5 - 4 = 1
2 - 1 = 1	5 - 5 = 0
2 - 2 = 0	5 - 6 = ___?
2 - 3 = ___?	5 - 7 = ___?

"I forgot to explain that when this problem presents itself, we must invent some new numbers. We think of 'above and below zero' or 'to the right and left of zero' as on a numberline or 'forward and backward from a starting point.'

"The numbers that seem to behave like counting numbers are called 'positive numbers' and those in the opposite direction from zero are called 'negative numbers.' We say that each positive number has an 'opposite' - a negative number. The opposite of positive six is negative six — the opposite of $^+6$ is $^-6$ and we say that

$^+a + {}^-a = 0$

"and in your examples, we say that

$^+2 - {}^+3 = {}^-1$, $^+5 - {}^+6 = {}^-1$, $^+5 - {}^+7 = {}^-2$

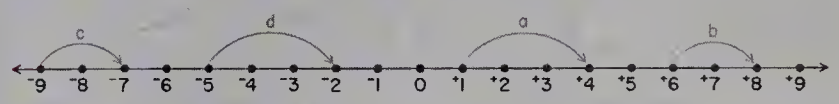
"With this start you ought to be able to extend each of the patterns in your two sets of examples in either direction. It may sound peculiar, but we say that since zero is not in any sense on either side of zero, we don't assign the name positive or negative to this number. It's simply 'zero'."

"That's right. So we have to invent more new numbers. If you had 2 sandwiches to divide among 3 Martians, how much would each get? We invent $\frac{2}{3}$ as the answer to that question."

"Almost as simple as that. If 2 Martians share 7 sandwiches, then each would get 7 halves or 3 whole sandwiches and one-half of another. We call that 3 and $\frac{1}{2}$."

(114) $a - b = x$ . . . Subtraction is always possible! . . . (The "principle of closure" for subtraction)

As X-Nighyun experiments with some of the new numbers — positive and negative numbers and zero — we ought to be prepared to answer the questions that will arise. We might explain to him what we mean by a number line.



We can simplify our language a little by referring to $^+2$ as "two" and $^-2$ as "the opposite of two." Of course, if you prefer, you can speak of "positive two" and of "negative two."

These new numbers behave on a number line much as the counting numbers do. We begin by thinking of addition as a jump to the right (see green arrows).

a. $^+1 + {}^+3 = {}^+4$ c. $^-9 + {}^+2 = {}^-7$

b. $^+6 + {}^+2 = {}^+8$ d. $^-5 + {}^+3 = {}^-2$

We can also begin thinking of subtraction as a jump in the opposite direction . . . if the known addend is not the opposite of a number.

e. $^+9 - {}^+2 = {}^+7$ g. $^+1 - {}^+2 = {}^-1$

f. $^+5 - {}^+1 = {}^+4$ h. $^-3 - {}^+5 = {}^-8$

other examples:

i. $^+2 + {}^+2 = {}^+4$ j. $^+2 - {}^+2 = 0$

k. $^-4 + {}^+3 = {}^-1$ l. $^-4 - {}^+3 = {}^-7$

m. $^+1 + {}^+8 = {}^+9$ n. $^+1 - {}^+8 = {}^-7$

Follow the Patterns!

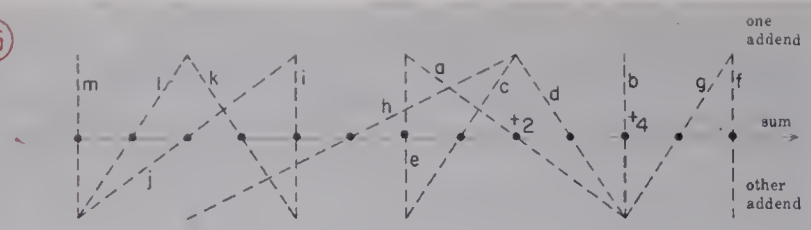
We know enough to get some patterns started. Where do the patterns lead?

I	II	III	IV
$^+3 + {}^+2 = {}^+5$	$^+3 - {}^+2 = {}^+1$	$^-3 + {}^+2 = {}^-1$	$^-3 - {}^+2 = {}^-5$
$^+3 + {}^+1 = {}^+4$	$^+3 - {}^+1 = {}^+2$	$^-3 + {}^+1 = {}^-2$	$^-3 - {}^+1 = {}^-4$
$^+3 + 0 = {}^+3$	$^+3 - 0 = {}^+3$	$^-3 + 0 = {}^-3$	$^-3 - 0 = {}^-3$
$^+3 + {}^-1 = {}^+2$	$^+3 - {}^-1 = {}^+4$	$^-3 + {}^-1 = {}^-4$	$^-3 - {}^-1 = {}^-2$
$^+3 + {}^-2 = {}^+1$	$^+3 - {}^-2 = {}^+5$	$^-3 + {}^-2 = {}^-5$	$^-3 - {}^-2 = {}^-1$
$^+3 + {}^-3 = 0$	$^+3 - {}^-3 = {}^+6$	$^-3 + {}^-3 = {}^-6$	$^-3 - {}^-3 = 0$

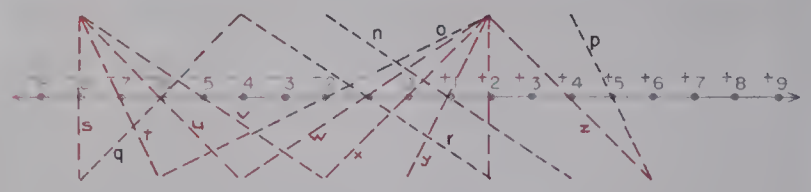
These results will help on the triple number line on the next page

Stretch your mental muscles.

x	2	4	8	$\frac{1}{2}$	.5	3	$\frac{1}{3}$	7	$\frac{1}{4}$	7	1	7	$\frac{3}{4}$
y	5	7	3	$\frac{1}{2}$	7	7	$\frac{1}{4}$	9	$\frac{1}{4}$	7	1	0	$\frac{3}{4}$
$x + y$	7	11	11	1	7.5	10	$\frac{7}{12}$	16	$\frac{1}{2}$	14	2	7	$1\frac{1}{2}$
$x \cdot y$ or xy	10	28	24	$\frac{1}{4}$	3.5	21	$\frac{1}{12}$	63	$\frac{1}{16}$	49	1	0	$\frac{9}{16}$



lines	a	b	c	d	e	f	g	h	i	j	k	l	m
one addend	0	+2	+1	+1	0	+3	+3	+1	-1	-1	-2	-2	-3
other addend	+2	+2	0	+2	0	+3	+2	-2	-1	-3	-1	-3	-3
sum	+2	+4	+1	+3	0	+6	+5	-1	-2	-4	-3	-5	-6



Draw lines (and label them) to find the missing addend or sum.

lines	n	o	p	q	r	s	t	u	v	w	x	y	z
one addend	-1	+1	+2	-2	-2	-4	-4	-4	-4	+1	+1	+1	+1
sum	+1	-2	+5	-6	-1	-8	-7	-6	-5	-1	0	+1	+4
other addend	+2	-3	+3	-4	+1	-4	-3	-2	-1	-2	-1	0	+3

Find the results in the examples below.
Use a straight-edge, but do not draw lines.

examples	A	B	C	D	E	F	G	H	I	J	K	L	M
sum	-1	-1	-1	-1	-1	0	0	0	+1	+1	+1	+1	+1
one addend	-3	-2	-1	0	+1	+2	0	-2	-1	0	+1	+2	+3
other addend	+2	+1	0	-1	-2	-2	0	+2	+2	+1	0	-1	-2

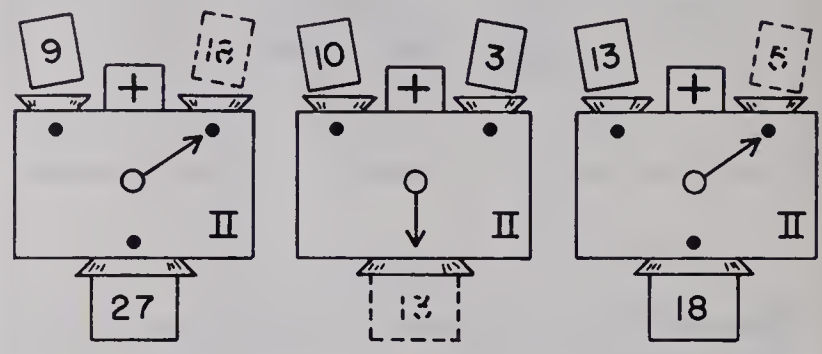
A record of the machine results would show:

$$2,\ 7 \rightarrow\ 9$$
$$8,\ 3 \rightarrow 11$$
$$9,11 \rightarrow 20$$

If an initial expression looks like this:

$$(27-9) = (10+3)$$

Machine II would be used in the following ways:



An experienced operator might summarize this result as:

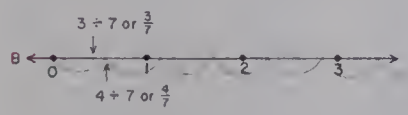
$$(27-9) - (10+3) \rightarrow 18 - (10+3)$$
$$\rightarrow 18 - 13$$
$$\rightarrow\ 5$$

Division is always possible! . . . (The "principle of closure" for division)

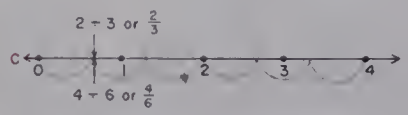
X-Nighyun knows about "rows and columns" and can find that $3 \times 7 = 21$ because he can arrange sandwiches so there are 3 rows and 7 columns . . . and count to 21. And, so, he knows that $21 \div 3 =$ ___ asks for a missing factor . . . 7 in this case.

But $7 \div 2 =$ ___ still bothers him because he doesn't know any answer to $2 \times$ ___ $= 7$.

Suppose we explain that we invent numbers to label points between those on the number line labeled with the numbers of counting.

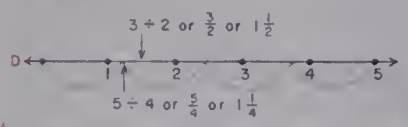


Divide the distance from 1 on the number line in 7 equal parts. This suggests the idea of $1 \div 7$ or $\frac{1}{7}$. Divide the distance from 2 in 7 parts. One of these suggests the idea of 2 divided by 7 or $\frac{2}{7}$ . . . etc.



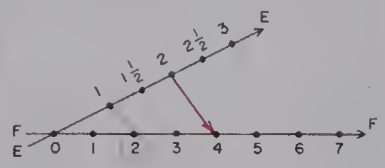
By this procedure, we sometimes find that points have more than one label; $2 \div 3$ and $4 \div 6$ name the same point.

Similarly, we can name points that are not between 0 and 1.



Suppose we tell X-Nighyun about an interesting machine he can use to see multiplication from a new viewpoint.

Draw two number lines that cross — and with "0" labeling the point of intersection.



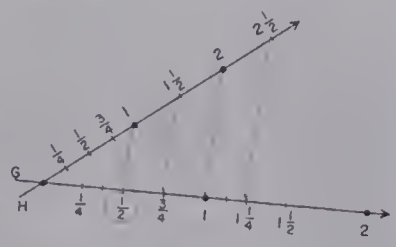
If you want to multiply by 2, for example, draw a line connecting the point labeled 2 on one number line (F) to the point labeled 1 on the other number line (E).

Next, draw a line from the point on line E labeled by any other factor — draw it parallel to the first line.

$2 \cdot 1\frac{1}{2} =$ 3 $\quad 2 \cdot 2\frac{1}{2} =$ 5

$2 \cdot 3 =$ 6 $\quad 2 \cdot 2 =$ 4

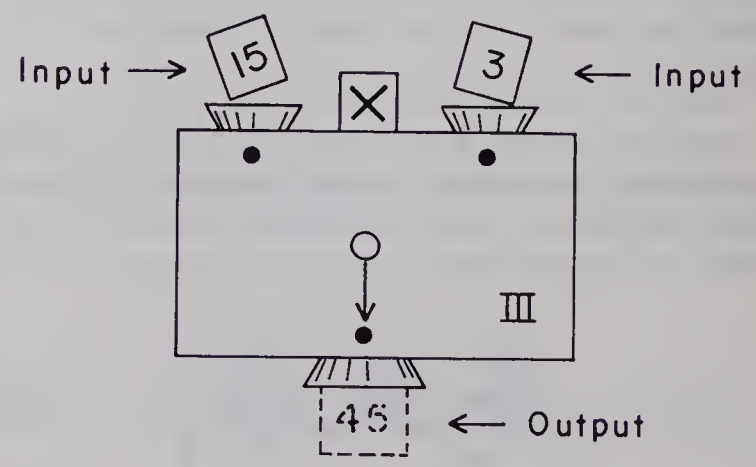
Here is another example. Notice that the angle between the lines has been changed and so has the scale on each line.



Suppose you want to multiply by $\frac{1}{2}$. Connect the point on G labeled $\frac{1}{2}$ to the point labeled 1 on H (solid green line). And, if you want to multiply, in other examples by $1\frac{1}{2}$, connect the point on G so labeled to the point labeled 1 on number line H (solid blue lines).

Machine II seems to be providing results for situations concerning addition and, with a switch of the pointer, providing results concerning subtraction. However, the machine accepts as individual inputs only certain names for numbers: names that do not include + or − . . . or × or ÷.

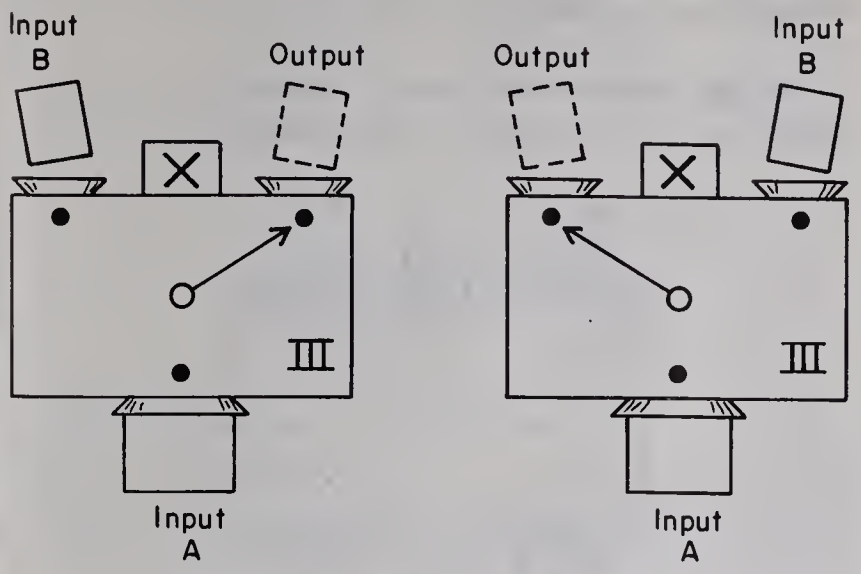
Machine III looks very similar to the first two. The result with 15 and 3 as inputs looks like this:



Other results are:

Inputs			Output
1/2 ,	7	⟶	3 1/2
.01 ,	.011	⟶	.00011
$\sqrt{4}$,	$\sqrt{25}$	⟶	10

With the pointer in alternate positions, the following results are noted:

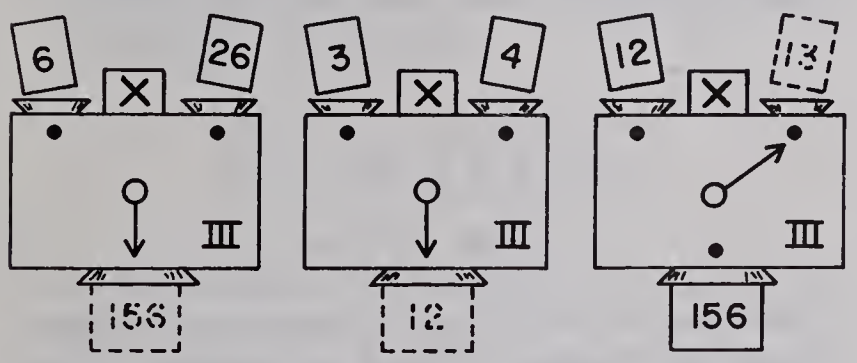


Inputs			Output
A	B		
39	13	⟶	3
1/2	1/3	⟶	1 1/2
.3	.5	⟶	.6
392	8	⟶	49

Experience shows that Machine III, like Machine II, will accept only certain names for numbers:

Accepts	Rejects
3.9	3 × 1.3
4/3	1 + 1/3
7/18	7 ÷ 18
84	90 − 6

The results of employing Machine III to help with the expression (6×26) ÷ (3×4) would look like this:

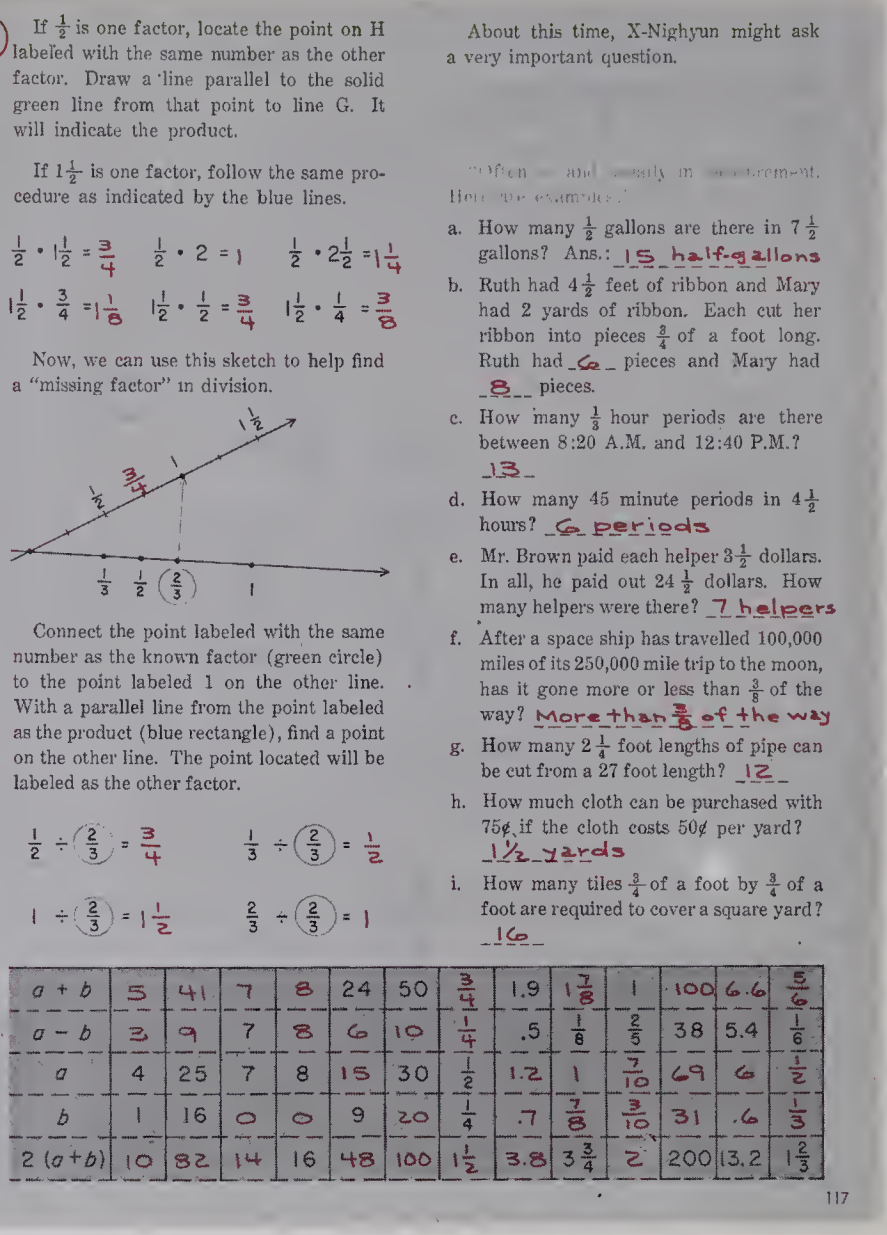


The next step might be:

$$(6\times26) \div (3\times4) \longrightarrow 156 \div (3\times4)$$
$$\longrightarrow 156 \div 12$$
$$\longrightarrow 13$$

Are Machines I, II and III consistent?

It would be unfortunate if operation of Machine I and Machine II or III led to contradictory results. If Machine I indicated that 10 + 3 is another name for 6 + 7, and Machine II did not indicate 10 + 3 → 13 and 6 + 7 → 13, we wouldn't have much use for them as computers. But, no matter how often we use these machines, neither Machine II nor Machine III ever contradicts Machine I.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If 1/2 is one factor, locate the point on H labeled with the same number as the other factor. Draw a line parallel to the solid green line from that point to line G. It will indicate the product.
If 1 1/2 is one factor, follow the same procedure as indicated by the blue lines.
1/2 • 1 1/2 = 3/4
1/2 • 2 = 1
1/2 • 2 1/2 = 1 1/4
1 1/2 • 3/4 = 1 1/8
1 1/2 • 1/2 = 3/4
1 1/2 • 1/4 = 3/8
Now, we can use this sketch to help find a "missing factor" in division.
Connect the point labeled with the same number as the known factor (green circle) to the point labeled 1 on the other line. With a parallel line from the point labeled as the product (blue rectangle), find a point on the other line. The point located will be labeled as the other factor.
1/2 ÷ 2/3 = 3/4
1/3 ÷ 2/3 = 1/2
1 ÷ 2/3 = 1 1/2
2/3 ÷ 2/3 = 1
About this time, X-Nighyun might ask a very important question.
a. How many 1/2 gallons are there in 7 1/2 gallons? Ans.: 15 half-gallons
b. Ruth had 4 1/2 feet of ribbon and Mary had 2 yards of ribbon. Each cut her ribbon into pieces 3/4 of a foot long. Ruth had 6 pieces and Mary had 8 pieces.
c. How many 1/3 hour periods are there between 8:20 A.M. and 12:40 P.M.? 13
d. How many 45 minute periods in 4 1/2 hours? 6 periods
e. Mr. Brown paid each helper 3 1/2 dollars. In all, he paid out 24 1/2 dollars. How many helpers were there? 7 helpers
f. After a space ship has travelled 100,000 miles of its 250,000 mile trip to the moon, has it gone more or less than 3/8 of the way? More than 3/8 of the way
g. How many 2 1/4 foot lengths of pipe can be cut from a 27 foot length? 12
h. How much cloth can be purchased with 75¢ if the cloth costs 50¢ per yard? 1 1/2 yards
i. How many tiles 3/4 of a foot by 3/4 of a foot are required to cover a square yard? 16
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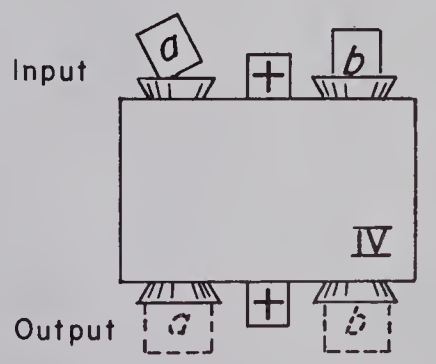
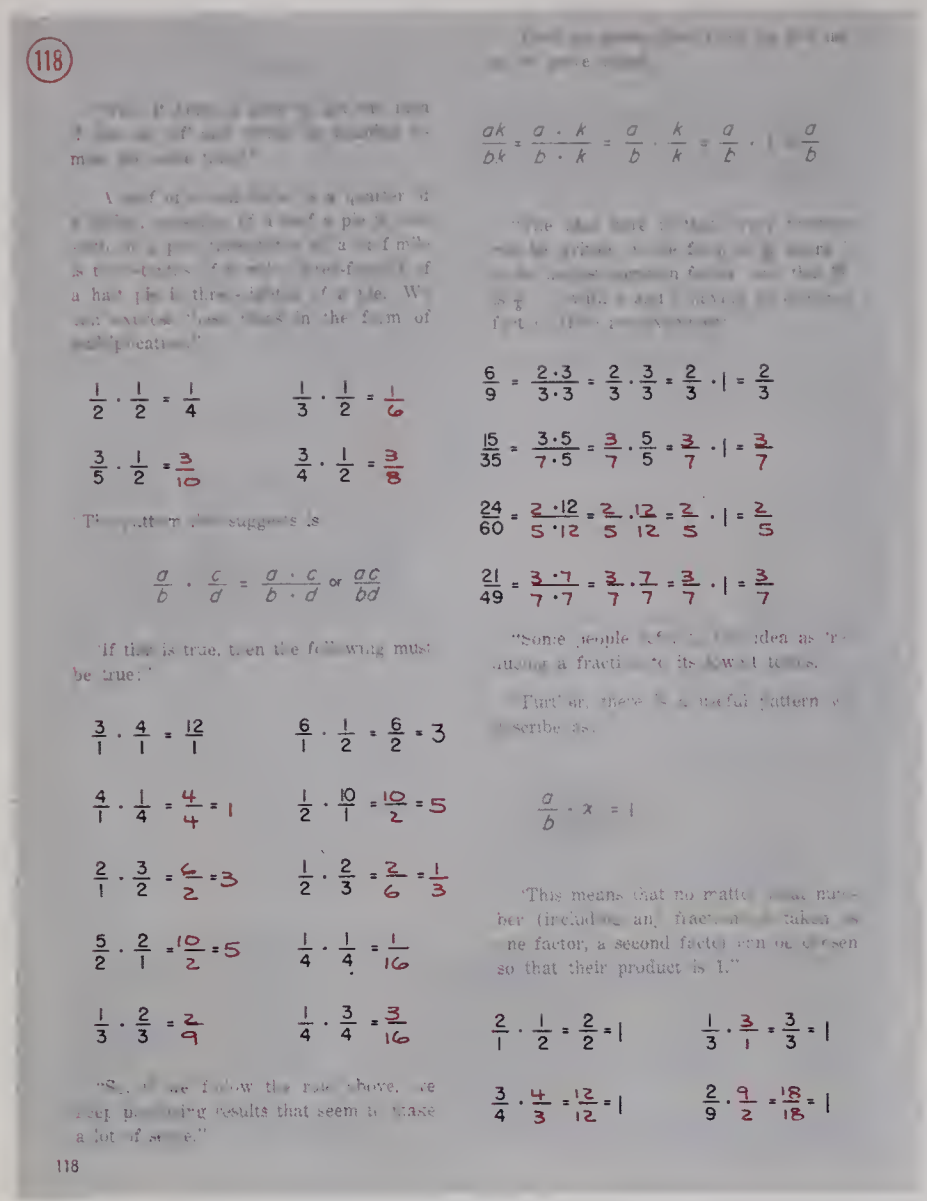
a + b	5	41	7	8	24	50	3/4	1.9	1 7/8	1	100	6.6	5/6
a − b	3	9	7	8	6	10	1/4	.5	1/8	2/5	38	5.4	1/6
a	4	25	7	8	15	30	1/2	1.2	1	7/10	69	6	1/2
b	1	16	0	0	9	20	1/4	.7	7/8	3/10	31	.6	1/3
2 (a+b)	10	82	14	16	48	100	1 1/2	3.8	3 3/4	2	200	13.2	1 2/3

More Machines

Machine IV introduces a different idea. Think of any two numbers. Write one of these numbers on one input card, the other on another. On the back of each card, make a mark so you can remember which is which (if they are different). Suppose you use these cards as inputs to Machine IV and that you labeled their backs *a* and *b*:



Suppose you tried this several times, each time labeling your cards differently so you could keep track. A record of the results might be: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$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
$\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$
$\frac{3}{5} \cdot \frac{1}{2} = \frac{3}{10}$
$\frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$
$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$ or $\frac{ac}{bd}$
$\frac{3}{1} \cdot \frac{4}{1} = \frac{12}{1}$
$\frac{6}{1} \cdot \frac{1}{2} = \frac{6}{2} = 3$
$\frac{4}{1} \cdot \frac{1}{4} = \frac{4}{4} = 1$
$\frac{1}{2} \cdot \frac{10}{1} = \frac{10}{2} = 5$
$\frac{2}{1} \cdot \frac{3}{2} = \frac{6}{2} = 3$
$\frac{1}{2} \cdot \frac{2}{3} = \frac{2}{6} = \frac{1}{3}$
$\frac{5}{2} \cdot \frac{2}{1} = \frac{10}{2} = 5$
$\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$
$\frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$
$\frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$
$\frac{ak}{bk} = \frac{a \cdot k}{b \cdot k} = \frac{a}{b} \cdot \frac{k}{k} = \frac{a}{b} \cdot 1 = \frac{a}{b}$
$\frac{6}{9} = \frac{2 \cdot 3}{3 \cdot 3} = \frac{2}{3} \cdot \frac{3}{3} = \frac{2}{3} \cdot 1 = \frac{2}{3}$
$\frac{15}{35} = \frac{3 \cdot 5}{7 \cdot 5} = \frac{3}{7} \cdot \frac{5}{5} = \frac{3}{7} \cdot 1 = \frac{3}{7}$
$\frac{24}{60} = \frac{2 \cdot 12}{5 \cdot 12} = \frac{2}{5} \cdot \frac{12}{12} = \frac{2}{5} \cdot 1 = \frac{2}{5}$
$\frac{21}{49} = \frac{3 \cdot 7}{7 \cdot 7} = \frac{3}{7} \cdot \frac{7}{7} = \frac{3}{7} \cdot 1 = \frac{3}{7}$
$\frac{a}{b} \cdot x = 1$
$\frac{2}{1} \cdot \frac{1}{2} = \frac{2}{2} = 1$
$\frac{1}{3} \cdot \frac{3}{1} = \frac{3}{3} = 1$
$\frac{3}{4} \cdot \frac{4}{3} = \frac{12}{12} = 1$
$\frac{2}{9} \cdot \frac{9}{2} = \frac{18}{18} = 1$

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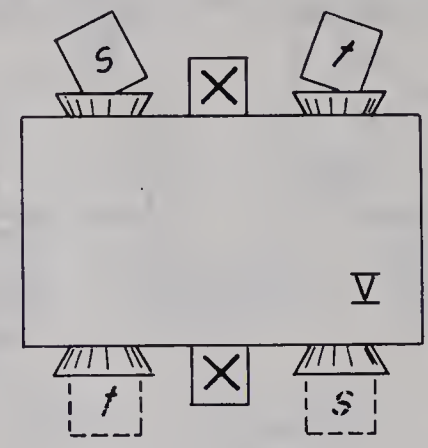
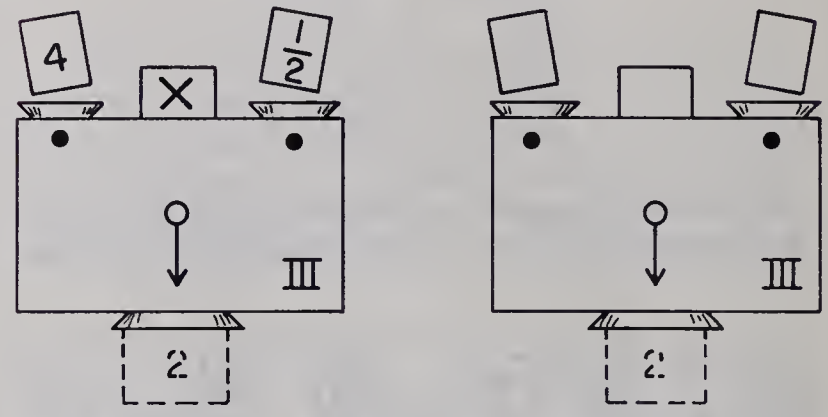
We are happy to find that the results of Machine IV and the results of Machine II do not lead to any disturbing results.

Similar experiments with Machine V—a new Machine—follow much the same pattern.



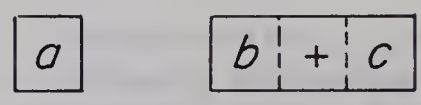
Results before turning cards over	Results after turning cards over
(1) s,t → t,s	*4, 1/2 → 1/2, 4*
(2) d,m → m,d	*.7, 1 → 1, .7*
(3) f,h → h,f	*2/3, 5/7 → 5/7, 2/3*

Each pair of cards on the right is fed into Machine III, and the results are:



2 → 2
.7 → .7
10/21 → 10/21

Machines VI and VII have a strong resemblance. To operate Machine VI, think of a number. Write it on a card; label the back, perhaps with the letter *a*. Now think of a pair of numbers; write each on a card; label the backs, perhaps with *b* and *c*. With another card, labeled +, tape them together into a single card so the two cards look like this:



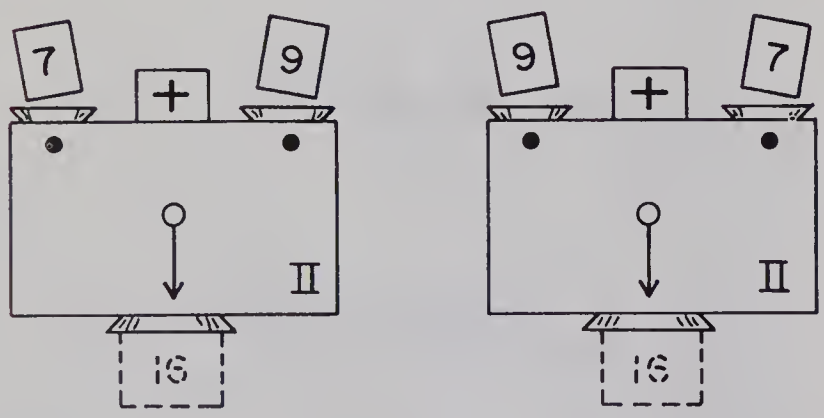
Used as inputs for Machine VI, the following is the result:

a, b → b, a
x, y → y, x
c, r → r, c

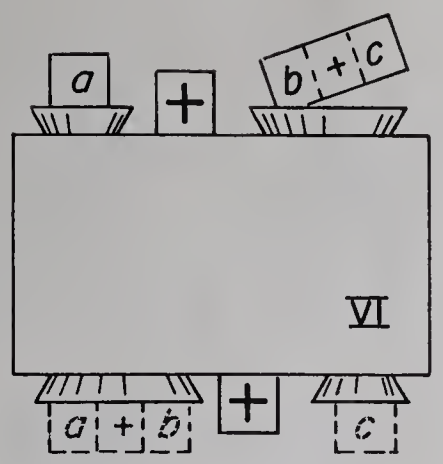
Turn over your cards, showing the numbers you wrote on them:

7, 9 → 9, 7
.1, 1/2 → 1/2, .1
3/7, 1/21 → 1/21, 3/7

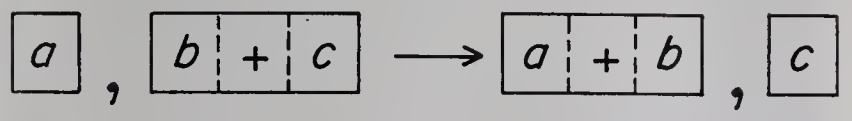
Now, in each case, we feed each pair into Machine II and keep track of the results:



16 → 16
.6 → .6
10/21 → 10/21



These results can be recorded as:



Using parentheses to indicate "cards taped together," we can write this as:

a, (b+c) → (a+b), c

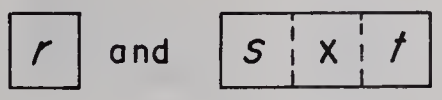
Other experiments, using other letters, might be recorded as:

f, (h+k) → (f+h), k
x, (y+z) → (x+y), z

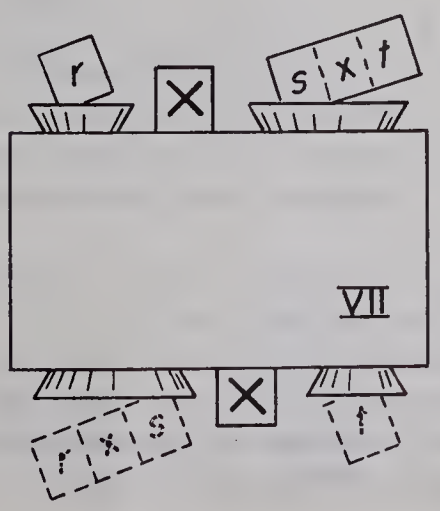
When the cards are turned over, the three experiments noted above might be recorded as:

7, (3+8) → (7+3), 8
1/2, (2/5+3) → (1/2+2/5), 3
.1, (1.5+.5) → (.1+1.5), .5

Machine VII follows a similar pattern. We think of numbers and note them on cards labeled as before, and the backs look like this:



Fed into Machine VII, we have the following results:



"I'll show you.

"Think of any division fact such as 3 ½ = ___. Now think of any number and multiply both the product and either factor by that number. Try 4, 8 and 20."

$3 \cdot 4 \div \frac{1}{2} \cdot 4 = 6$ or $3 \cdot 4 \div \frac{1}{2} = 6 \cdot 4$

$12 \div 2 = 6$ or $12 \div \frac{1}{2} = 24$

$3 \cdot 8 \div \frac{1}{2} \cdot 8 = 6$ or $3 \cdot 8 \div \frac{1}{2} = 6 \cdot 8$

$24 \div 4 = 6$ or $24 \div \frac{1}{2} = 48$

$3 \cdot 20 \div \frac{1}{2} \cdot 20 = 6$ or $3 \cdot 20 \div \frac{1}{2} = 6 \cdot 20$

$60 \div 10 = 6$ or $60 \div \frac{1}{2} = 120$

That is right. But let's use the pattern to see what sense we can make out of

$\frac{7}{8} \div \frac{2}{3} = ?$

"Will you agree with me that

$\frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \cdot x \div \frac{2}{3} \cdot x$?"

"All right. I'm going to select $\frac{3}{2}$ as x and I will write

$\frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \cdot \frac{3}{2} \div \frac{2}{3} \cdot \frac{3}{2}$ "

"Because $\frac{2}{3} \cdot \frac{3}{2} = \frac{6}{6} = 1$, and I can now write

$\frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \cdot \frac{2}{3} \div 1 = \frac{7}{8} \cdot \frac{2}{3}$

because division of any number by 1 leaves the number unchanged. Now, I use the pattern

$\frac{a}{b} \cdot \frac{x}{y} = \frac{ax}{by}$

and I write

$\frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \cdot \frac{3}{2} = \frac{21}{16} = 1\frac{5}{16}$ "

$\frac{6}{1} \div \frac{1}{2} = \frac{6}{1} \cdot \frac{2}{1} \div \frac{1}{2} \cdot \frac{2}{1} = \frac{6}{1} \cdot \frac{2}{1} \div 1 = \frac{6}{1} \cdot \frac{2}{1} = 12$

$\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \cdot \frac{4}{1} = \frac{4}{2} = 2$

$\frac{3}{4} \div \frac{1}{8} = \frac{3}{4} \cdot \frac{8}{1} = \frac{24}{4} = 6$

$\frac{4}{2} \div \frac{2}{3} = \frac{4}{2} \cdot \frac{3}{2} = \frac{12}{4} = 3$

$\frac{2}{3} \div \frac{7}{8} = \frac{2}{3} \cdot \frac{8}{7} = \frac{16}{21}$

 $x + y = z$ Addition is always possible.

Examples given by X-Nighyun:

$28 + 35 =$ $3.5 + .72 =$ $\frac{2}{3} + \frac{7}{8} =$

$^{-}3 + ^{-}4 =$ $^{-}3 + ^{+}4 =$ $^{+}3 + ^{-}4 =$

"Since numbers have many names, we can write 28 + 35 as follows:

20 + 8 + 30 + 5

"We select the terms 8 and 30 and then apply the Commutative principle:

20 + (8 + 30) + 5 = 20 + (30 + 8) + 5

"Next, we can apply the Associative principle to the first three terms:

20 + (30 + 8) + 5 = (20 + 30) + 8 + 5

"Then we find other names for 20 + 30 and for 8 + 5 and write:

(20 + 30) + 8 + 5 = 50 + 13

"We find another name for 13 and use the Associative rule again:

50 + (10 + 3) = (50 + 10) + 3 = 60 + 3 = 63

"Of course, we shorten this long business and think of what's happening. We look at it this way:"

28	20 + 8	50 + 0
35	30 + 5	10 + 3
63	50 + 13	60 + 3 = 63

Try some examples of addition using the form suggested here.

"The next two examples are essentially the same if you write them both as addition of common fractions.

$\frac{35}{10} + \frac{72}{100} =$ and $\frac{2}{3} + \frac{7}{8} =$

"We want common denominators so we multiply certain names for one to accomplish our purpose."

$\frac{35}{10} \cdot \frac{10}{10} + \frac{72}{100} =$

$\frac{350}{100} + \frac{72}{100} = \frac{422}{100}$

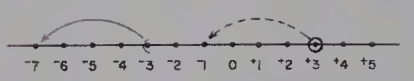
$= 4.22$

$\frac{2}{3} \cdot \frac{8}{8} + \frac{7}{8} \cdot \frac{3}{3} =$

$\frac{16}{24} + \frac{21}{24} = \frac{37}{24}$

$= 1\frac{13}{24}$

"The next three examples are related. Think of a number line. In each example, start at the point labeled as the first addend. Then take a jump as indicated by the other addend — but the direction is indicated by the sign: positive to the right, negative to the opposite direction."



$^{-}3 + ^{-}4 = ^{-}7$ $^{+}1$

$^{+}3 + ^{-}4 = ^{-}1$ $^{+}3 + ^{+}4 = ^{+}7$

$^{-}4 + ^{+}3 = ^{-}1$ $^{+}4 + ^{+}3 = ^{+}7$

$^{-}2 + ^{-}1 = ^{-}3$ $^{-}2 + ^{+}1 = ^{-}1$

$^{-}5 + ^{+}5 = 0$ $^{+}3 + ^{-}3 = 0$

m	48	$\frac{1}{2}$	2	12	3	.2	17	3	1	3	119	$\frac{1}{6}$	$\frac{7}{8}$
n	4	$\frac{1}{2}$	$\frac{1}{3}$	2	$\frac{3}{4}$	.4	6	5	.1	6	17	$\frac{1}{3}$	$\frac{2}{3}$
$m \div n$	12	1	6	6	4	.5	$2\frac{5}{6}$	$\frac{3}{5}$	10	$\frac{1}{2}$	7	$\frac{1}{2}$	$1\frac{5}{16}$
$2mn$	384	$\frac{1}{2}$	$1\frac{1}{3}$	48	$4\frac{1}{2}$	.16	204	30	.2	36	4,046	$\frac{1}{9}$	$1\frac{1}{6}$
$2n + m$	56	$1\frac{1}{2}$	$2\frac{2}{3}$	16	$4\frac{1}{2}$	1	29	13	1.2	15	153	$\frac{5}{6}$	$2\frac{5}{24}$

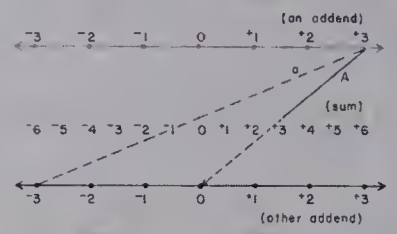
(122) $p - q = r$ Subtraction is always possible.

Examples submitted by X-Nighyun:

$53 - 17 =$ ____ $\quad 4 - 7 =$ ____

$^{+}9 - {}^{-}7 =$ ____ $\quad {}^{-}3 - {}^{+}8 =$ ____

A	$^{+}3 - {}^{+}3 = 0$	a	$^{+}3 + {}^{-}3 = 0$
B	$^{+}3 - {}^{+}2 = 1$	b	$^{+}3 + {}^{-}2 = 1$
C	$^{+}3 - {}^{+}1 = 2$	c	$^{+}3 + {}^{-}1 = 2$
D	$^{+}3 - 0 = 3$	d	$^{+}3 + 0 = 3$
E	$^{+}3 - {}^{-}1 = 4$	e	$^{+}3 + {}^{+}1 = 4$
F	$^{+}3 - {}^{-}2 = 5$	f	$^{+}3 + {}^{+}2 = 5$
G	$^{+}3 - {}^{-}3 = 6$	g	$^{+}3 + {}^{+}3 = 6$



$^{+}a - {}^{+}b =$ ____ $\quad {}^{+}a - {}^{-}b =$ ____

$^{-}a - {}^{+}b =$ ____ $\quad {}^{-}a - {}^{-}b =$ ____

H	$^{-}3 - {}^{+}3 = {}^{-}6$	h	$^{-}3 + {}^{-}3 = {}^{-}6$
I	$^{-}3 - {}^{+}2 = {}^{-}5$	i	$^{-}3 + {}^{-}2 = {}^{-}5$
J	$^{-}3 - {}^{+}1 = {}^{-}4$	j	$^{-}3 + {}^{-}1 = {}^{-}4$
K	$^{-}3 - 0 = {}^{-}3$	k	$^{-}3 + 0 = {}^{-}3$
L	$^{-}3 - {}^{-}1 = {}^{-}2$	l	$^{-}3 + {}^{+}1 = {}^{-}2$
M	$^{-}3 - {}^{-}2 = {}^{-}1$	m	$^{-}3 + {}^{+}2 = {}^{-}1$
N	$^{-}3 - {}^{-}3 = 0$	n	$^{-}3 + {}^{+}3 = 0$

$$^{+}a - {}^{+}b = {}^{+}a + {}^{-}b$$
$$^{+}a - {}^{-}b = {}^{+}a + {}^{+}b$$
$$^{-}a - {}^{+}b = {}^{-}a + {}^{-}b$$
$$^{-}a - {}^{-}b = {}^{-}a + {}^{+}b$$

(124) $n \cdot c = h$ Multiplication is always possible.

X-Nighyun had only a limited question about this matter.

$^{-}2 \cdot {}^{-}5 = ?$

$^{+}2 \cdot {}^{+}5 = {}^{+}10$

$2 \cdot {}^{-}5 = {}^{-}10$

$^{+}2 \cdot {}^{-}5 = {}^{-}10$

$^{-}2 \cdot 5 = {}^{-}10$ or $5 \cdot {}^{-}2 = {}^{-}10$

$^{-}2 \cdot 5 = 5 \cdot {}^{-}2$

$^{+}2 \cdot {}^{+}3 = {}^{+}6$	$^{-}2 \cdot {}^{+}3 = {}^{-}6$		
$^{+}2 \cdot {}^{+}2 = {}^{+}4$	$^{-}2 \cdot {}^{+}2 = {}^{-}4$		
$^{+}2 \cdot {}^{+}1 = {}^{+}2$	$^{-}2 \cdot {}^{+}1 = {}^{-}2$		
$^{+}2 \cdot 0 = 0$	$^{-}2 \cdot 0 = 0$		
$^{+}2 \cdot {}^{-}1 = {}^{-}2$	$^{-}2 \cdot {}^{-}1 = {}^{+}2$		
$^{+}2 \cdot {}^{-}2 = {}^{-}4$	$^{-}2 \cdot {}^{-}2 = {}^{+}4$		
$^{+}2 \cdot {}^{-}3 = {}^{-}6$	$^{-}2 \cdot {}^{-}3 = {}^{+}6$		

$^{-}2 \cdot {}^{-}5 = {}^{+}10$

$^{-}a \cdot {}^{-}b = {}^{+}ab$ ''

a	$^{-}1$	$^{+}4$	$^{+}2$	$^{+}5$	$^{-}5$	$^{+}5$	$^{-}5$	$^{+}6$	$^{+}18$	$^{+}\frac{1}{2}$	$^{+}.1$	$^{-}3$	$^{+}3$
b	$^{+}2$	$^{-}4$	$^{-}1$	$^{-}5$	$^{+}5$	$^{+}5$	$^{-}5$	$^{+}9$	0	$^{+}\frac{1}{2}$	$^{+}.1$	$^{+}13$	$^{-}2$
$a+b$	$^{+}1$	0	$^{+}1$	0	0	$^{+}10$	$^{-}10$	$^{+}15$	$^{+}18$	$^{+}1$	$^{+}.2$	$^{+}10$	$^{+}1$
ab	$^{-}2$	$^{-}16$	$^{-}2$	$^{-}25$	$^{-}25$	$^{+}25$	$^{+}25$	$^{+}54$	0	$^{+}\frac{1}{4}$	$^{+}.01$	$^{-}39$	$^{-}6$
$a-b$	$^{-}3$	$^{+}8$	$^{+}3$	$^{+}10$	$^{-}10$	0	0	$^{-}3$	$^{+}18$	0	0	$^{-}16$	$^{+}5$

(123)

With a similar interpretation, write about the situation suggested by the following headlines after they are completed:

(A,a) $^{+}3 - {}^{+}3 = 0$ and $^{+}3 + {}^{-}3 = 0$

(A,a) $^{+}3 - {}^{+}3 = {}^{+}3 + {}^{-}3$

(G,g) $^{+}3 - {}^{-}3 = {}^{+}3 + {}^{+}3$

(H,h) $^{-}3 - {}^{+}3 = {}^{-}3 + {}^{-}3$

(N,n) $^{-}3 - {}^{-}3 = {}^{-}3 + {}^{+}3$

$^{+}3 - {}^{+}3$...You earn $3 and you get rid of $3. You are $0 to the good.

$^{+}3 - {}^{-}3$...You earn $3 and you get rid of a $3 debt. You are $6 to the good.

$^{-}3 - {}^{+}3$.. You incur a $6 debt and get rid of $3. You are worse off by $6.

$^{-}3 - {}^{-}3$...You incur a debt of $3 and get rid of a $3 debt. You are $0 to the good.

$^{+}4 + {}^{+}5 = {}^{+}9$ I earned $4 and received $5 more. I am $9 to the good

$^{+}4 - {}^{+}7 = {}^{-}3$ I received $4 and spent $7 I am $3 worse off than before

$^{-}8 + {}^{-}10 = {}^{-}18$ I incurred a debt of $8 and another debt of $10 I am $18 worse off.

$^{-}15 + {}^{+}21 = {}^{+}6$ I incurred a debt of $15 and earned $21 I am $6 to the good

$^{-}18 - {}^{-}8 = {}^{-}10$ I incurred a debt of $18 and then got rid of one for $8 I am only $10 worse off

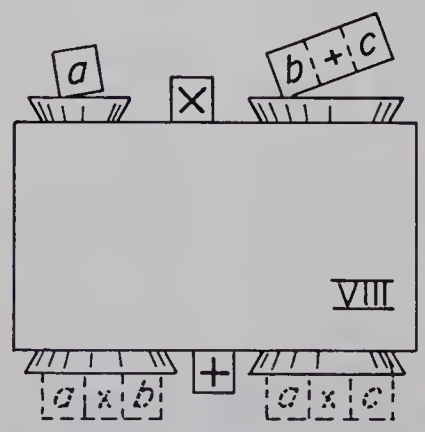
$^{-}1 - {}^{-}1 = 0$ I incurred a debt of $1 and got rid of another for $1. I'm right where I started

$^{+}\frac{1}{2} + {}^{-}\frac{1}{2} = 0$ I earned a half-dollar and incurred a half-dollar debt I'm right where I started

$^{-}\frac{3}{4} + {}^{-}\frac{1}{2} = {}^{-}1\frac{1}{4}$ After incurring a debt of 75¢ and another of 50¢ I'm 1\frac{1}{4}$ worse off

It is quite clear what the results of other experiments with Machine VII would produce.

M VIII (short for Machine VIII) may seem to produce disturbing results at first. To operate, prepare cards exactly as with MVI. Feed into MVIII. The results would look like this:



Using parentheses as before, this can be recorded as:

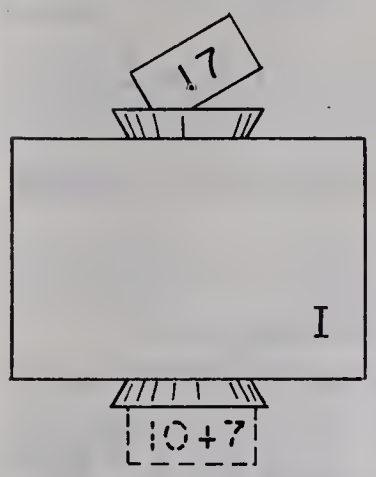
$$a, (b+c) \rightarrow (a \times b), (a \times c)$$

Other experiments could be tried and the cards turned over to reveal the numbers that had been thought of and written on the back:

$7, (4+6) \rightarrow (7\times4), (7\times6)$
$1/2, (2/3+1/3) \rightarrow (1/2\times2/3), (1/2 \times 1/3)$

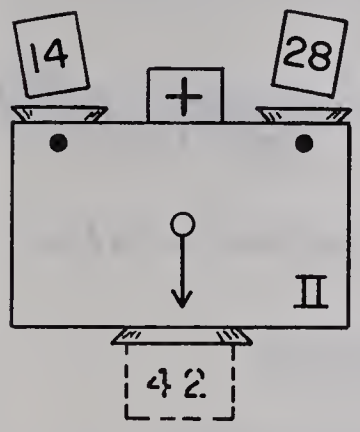
Let's look at all the machines and then, according to the number of inputs and outputs, consider some examples of results. And, instead of the commas, we will note the marking on the machine. And, instead of the arrow, we will write "=" to separate the input and the output.

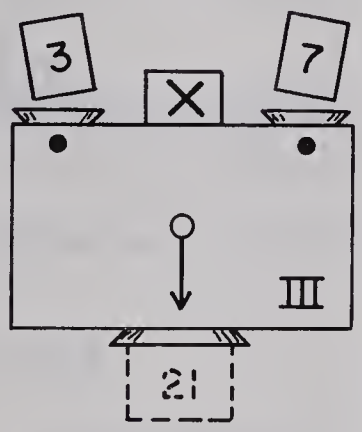
One input and one output



$17 = 10 + 7$
$12 = 3 \times 4$
$3 + 10 = 8 + 5$

Two inputs and one output





$14 + 28 = 42$
$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

$3 \times 7 = 21$
$.8 \times 1.1 = .88$

These machines are hand computers. M II can help us find the number that is the sum of any pair of numbers taken as addends, and M III can help us find the number that is the product of any pair of numbers taken as factors. (By changing the pointer, M II can help us find the number for a missing addend if the sum and one addend is given; and M III can help us find a missing factor if a product and one of the factors is given.)

These machines handle pairs of cards with a particular number written on each. Their output in each case is a card with a particular number written on it—and no names can contain the symbols for addition, multiplication, subtraction or division (except as a fractional form indicating division).

125

$r \div m = q$ ____ Division is always possible.

X-Nighyun suggested that he try to answer some of his own questions concerning this claim.

$^-6 \div {}^-2 =$ ___ or $24 \div {}^-3 =$ ___

$^-6 \div {}^-2 =$ ___ is the same as $^-2 \cdot$ ___ $= {}^-6$

$24 \div {}^-3 =$ ___ is the same as $^-3 \cdot$ ___ $= 24$

$^-2 \cdot$ $= {}^-6$ and $^-3 \cdot$ $= 24$

"Congratulation. What you've done certainly sounds reasonable to me.

"Now, using any sensible explanation you like, describe situations that fit the idea of the headlines in the next column. Complete the headline first."

$^+30 + {}^-42 = {}^-12$ From a boat on a lake I threw a stone 30 ft. in the air. It fell 42 feet to the bottom of the lake. The water was 12 feet deep.

$^-7 + {}^-6 = {}^-13$ I walked down 7 stairs and then down 6 more. I was then 13 stairs below my starting place.

$^+7 \cdot {}^-12 = {}^-84$ Each day of the week I spent 12¢. By the end of the week I spent 84¢.

$15 \div \frac{2}{3} = 22\frac{1}{2}$ My steps are $\frac{2}{3}$ of a yard long. How many steps do I take for each 15 yards?
Ans: $22\frac{1}{2}$ steps for 15 yards.

$^+7 + {}^-5 = {}^+2$ I earned ¢7 and spent ¢5. I had ¢2 left.

x	4	$\frac{1}{2}$	.7	$\frac{1}{2}$	15	10	19	$^+9$	2	$1\frac{1}{2}$	.8	$^+5$
y	5	$\frac{1}{8}$	1	$\frac{1}{3}$	33	1	57	$^-7$	3	10	1	$^-3$
z	6	$\frac{1}{4}$	9.2	$\frac{1}{4}$	77	.1	173	$^-13$	5	$9\frac{1}{2}$	1.5	$^+3$
$x + y$	9	$\frac{5}{8}$	1.7	$\frac{5}{6}$	48	11	76	$^+2$	5	$11\frac{1}{2}$	1.8	$^+2$
$y + z$	11	$\frac{3}{8}$	10.2	$\frac{7}{12}$	110	1.1	230	$^-20$	8	$19\frac{1}{2}$	2.5	0
$x + z$	10	$\frac{3}{4}$	9.9	$\frac{3}{4}$	92	10.1	192	$^-4$	7	11	2.3	$^+8$

125

126

You shouldn't be surprised, one day, if X-Nighyun started asking some strange questions. Like these:

Answer: 375,136,000 glooms

Answer: 7,500 gluds

Answer: 22,500 gluds

"But how can I tell when I don't know what a glooм is?"

Answer: 11.1 chorns

"Oh, you want to know how many glooms it is from you to Astra A?"

Answer: 25.9 chorns

"The distance must be about 375,000,000 glooms."

(75×7.4=555)
Answer: 75 space suits

Answer: 3.4 gluds.

Answer: 136,000 glooms

Answer: 6.8 gluds

Answer: 18,750 gluds

Answer: 186,500 glooms

126

Rockton — 595 miles — Junction City — 350 miles — Newton

Here are your questions for X-Nighyun.

1. "R[illegible] and S[illegible] are [illegible] 'S' left Rockton — [illegible] P[illegible] and [illegible] 45 miles per hour. How long will it take for S to reach Newton?"

21 hours

2. "A fast freight train — F for short — leaves from Newton and travels at the rate of 60 miles per hour. How long will it take to reach Rockton?"

15 3/4 hrs. (or 15 hrs. & 45 min.)

3. "If S left Rockton at 1 P.M., what time did it arrive at Newton?"

8. [illegible]

315 miles

9. [illegible]

315 miles

10. [illegible]

10 P.M. (after 9 hours)

11. [illegible]

630 miles

12. [illegible]

45 miles

13. [illegible]

270 miles

14. [illegible]

About 590 miles

15. "Of course, [illegible] some other station. A train, [illegible] City is 245 miles further from Rockton than from Newton. Now how far [illegible] from Junction City to Newton."

350 miles

16. "Now, [illegible] at 1 P.M. and [illegible] 45 miles per hour. [illegible] City and [illegible] 60 miles per hour. At what time did F leave Newton?"

at about 8:23 P.M.

Always more explaining to do. Could you make X-Nighyun understand our system of talking about time?

4. "All right, I'll change my question. If S left from Rockton and F from Newton at the same time how much earlier would F reach Rockton than S reached Newton?"

5 1/4 hrs.

5. "How far would S be from Newton when F reached Rockton?"

236 1/4 miles (5 1/4 · 45)

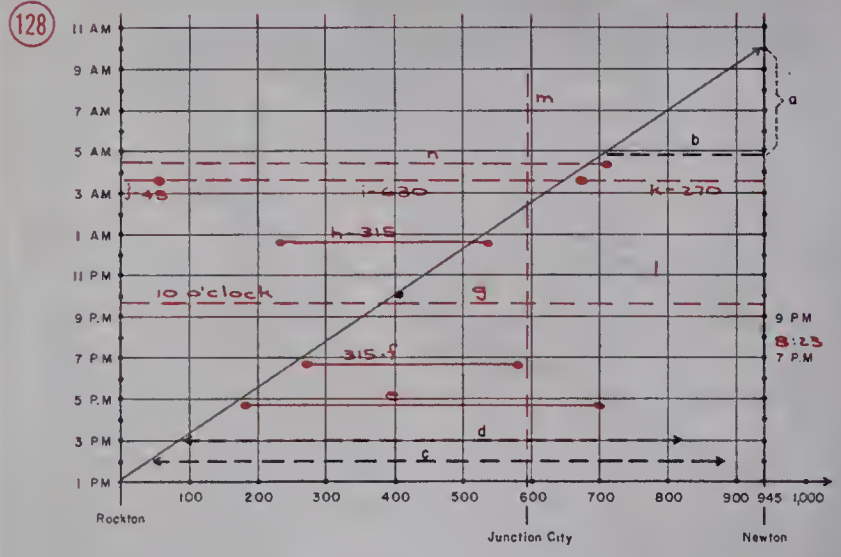
"S left from Rockton and F left from Newton at 1 P.M. on the same day. Now I have several questions to ask."

6. "How far apart would they be at 2 P.M.?"

840 miles apart

7. "How far apart at 5 P.M.?"

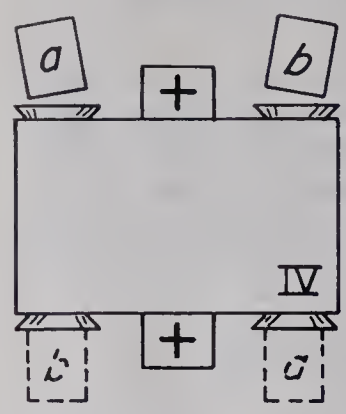
525 miles apart



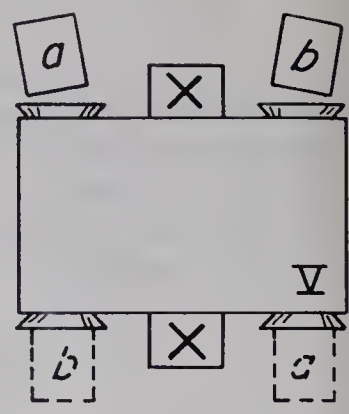
By interplanetary mail, X-Nighyun sent a note along with the sketch above.

Can you discover ways in which X-Nighyun used his sketch?

Two outputs and two inputs



$a + b = b + a$

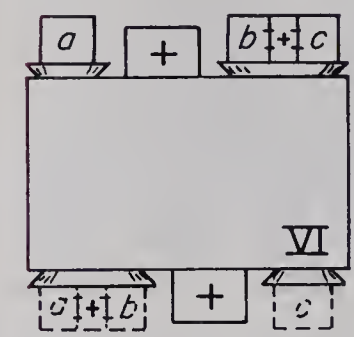


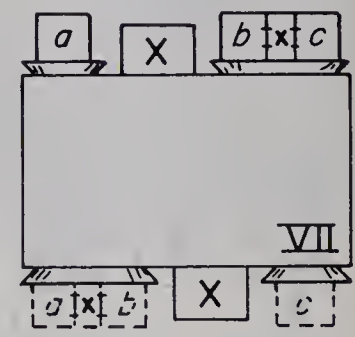
$a \times b = b \times a$

As cards are turned over, experiments recorded might be:

3+7=7+3
397+129=129+397
1/2+3/10=3/10+1/2

8×4=4×8
1.3×29/37=29/37×1.3
1 8/12×50/100=50/100×1 8/12



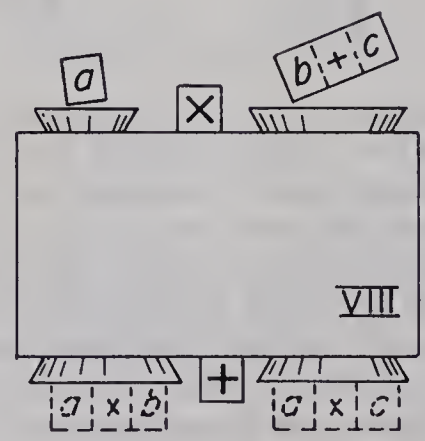


$a + (b+c) = (a+b)+c$ $\qquad$ $a \times (b \times c) = (a \times b) \times c$

Backs of experiments

4+(6+10)=(4+6)+10
.1+(.9+8)=(.1+.9)+8

2×(3×5)=(2×3)×5
1/2×(1/3×1/4)=(1/2×1/3)×1/4



$a \times (b+c) = (a \times b) + (a \times c)$

Back of experiments

$2 \times (b+c) = (2 \times 10) \times (2 \times 8)$
$2/3 \times (.7+4) = (2/3 \times .7) + (2/3 \times 4)$

Depending upon the familiarity of the teacher with developments such as this, there are unlimited examples of an interesting game we can play with these machines. Suppose we want to find out if we can change on expression into another by using the machines to provide each. An example:

Can we change: a + (b+c) to (b+a) + c?

On the left, we indicate the way we use the machine – write the results – and indicate the machine used:

$a + (b + c)$ goes to $(a + b) + c$ by M VI

Then

$(a + b) + c$ goes to $(b + a) + c$ by M IV

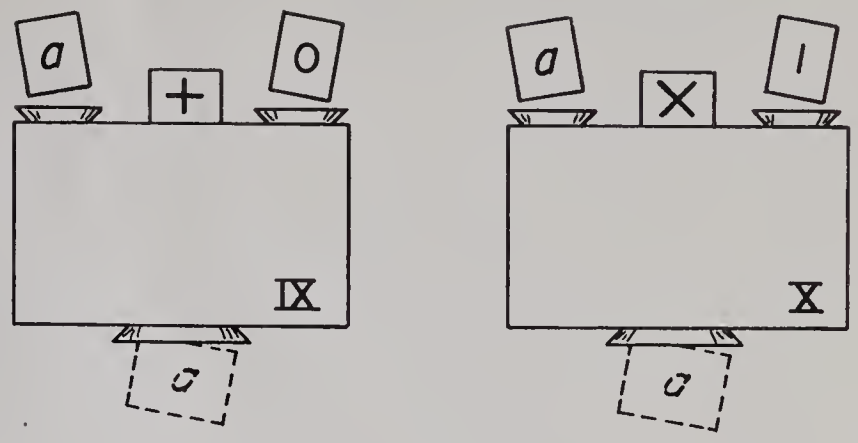
So, a + (b+c) was changed to (b+a) + c . . . which was our goal in this example.

Can we change a × (b+c) to (a×c) + (a×b)?

$a \times (b + c)$ goes to $a \times (c + b)$ by M IV

$a \times (c + b)$ goes to $(a \times c) + (a \times b)$ by M VIII

You may want to introduce two more machines:



Further, we try each machine "in reverse," and we find that all give reasonable results.

Because of the way these machines operate, we might name them:

M I	"Numbers have many names"
M II	"Number adder" (subtracter, when reversed)
M III	"Number multiplier" (divider, when reversed)
M IV	"Commuter for addition"
M V	"Commuter for multiplication"
M VI	"Associater for addition"
M VII	"Associater for multiplication"
M VIII	"Distributer" (of multiplication over addition)
M IX	"Unchanger" for addition
M X	"Unchanger" for multiplication

Teacher Observations and Ideas:

Teacher's Observations and Ideas:

Teacher's Observations and Ideas:

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